E9 205 Machine Learning for Signal Processing

EM for Gaussian Mixture Model

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Gaussian Mixture Models

A Gaussian Mixture Model (GMM) is defined as

$$p(\mathbf{x}|\boldsymbol{\Theta}) = \sum_{k=1}^{K} \alpha_k p(\mathbf{x}|\boldsymbol{\theta}_k)$$

$$p(\mathbf{x}|\boldsymbol{\theta}_k) = \frac{1}{\sqrt{(2\pi)^D |\boldsymbol{\Sigma}_k|}} exp\left\{ -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_k)^* \boldsymbol{\Sigma}_k^{-1} (\mathbf{x}-\boldsymbol{\mu}_k) \right\}$$

The weighting coefficients have the property

$$\sum_{k=1}^{K} \alpha_k = 1$$

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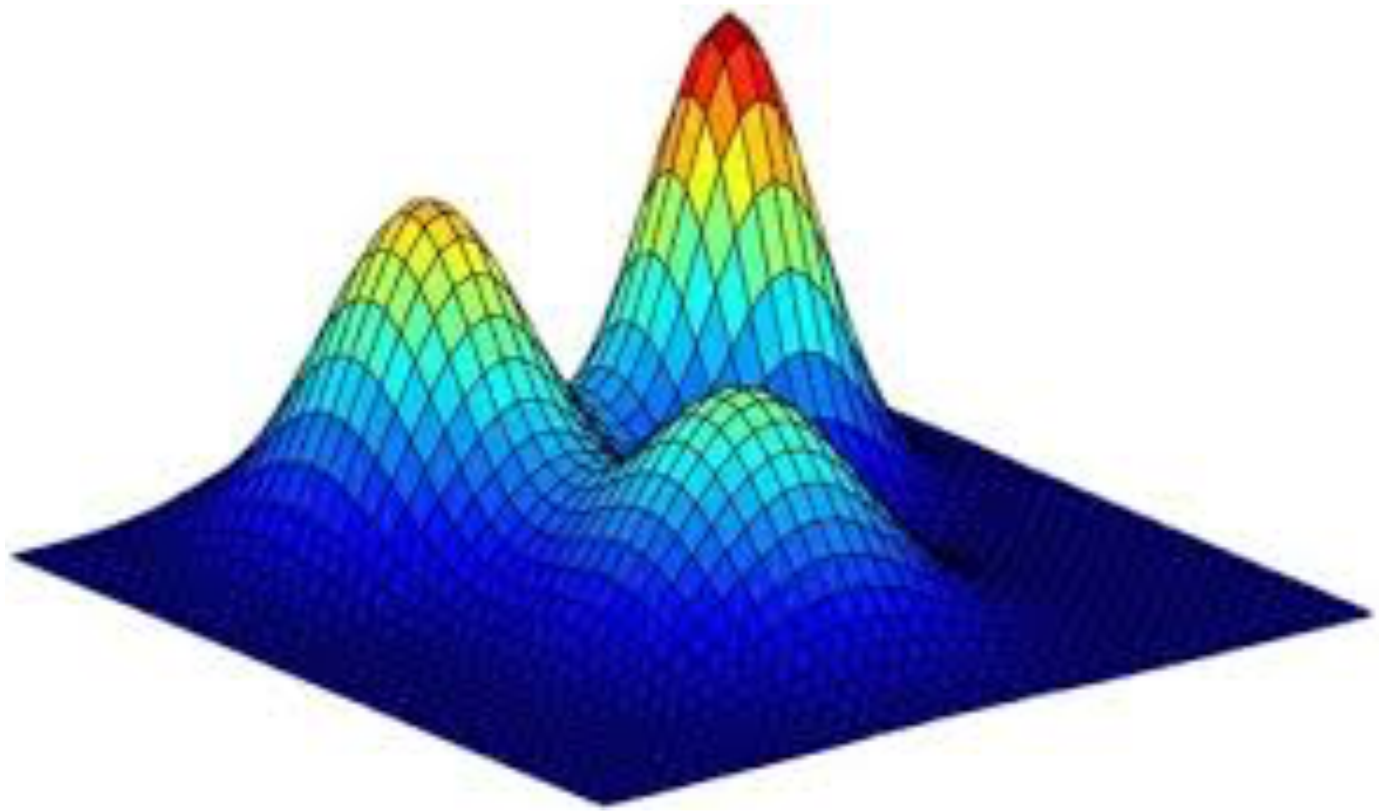
Gaussian Mixture Models

- Properties of GMM
  - Can model multi-modal data.
  - Identify data clusters.
  - Can model arbitrarily complex data distributions



The set of parameters for the model are

$$\boldsymbol{\Theta}_k = \{\alpha_k, \boldsymbol{\theta}_k\}_{k=1}^{K} \qquad \boldsymbol{\theta}_k = \{\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k\}$$

The number of parameters is $KD^2 + KD + K$

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MLE for GMM

- The log-likelihood function over the entire data in this case will have a logarithm of a summation

$$\log L(\boldsymbol{\Theta}) = \sum_{i=1}^{N} \log \left(\sum_{k=1}^{K} \alpha_k p(\mathbf{x}_i | \boldsymbol{\theta}_k) \right)$$

- Solving for the optimal parameters using MLE for GMM is not straight forward.

- Resort to the Expectation Maximization (EM) algorithm

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Roadmap

- Motivation
- Types of Classifiers
- Generative Modeling
- Gaussian Models and Maximum Likelihood Estimation
- Gaussian Mixture Models
- EM Algorithm for Parameter Estimation
- Hidden Markov Models
- Extensions and Applications
- Summary

Expectation Maximization Algorithm

- Iterative procedure.
- Assume the existence of hidden variable $\mathbf{z}_i$ associated with each data sample $\mathbf{x}_i$
- Let the current estimate (at iteration n) be $\boldsymbol{\Theta}^n$ Define the Q function as

$$
\begin{aligned}
Q(\boldsymbol{\Theta}, \boldsymbol{\Theta}^n) &= E_{\mathbf{z}|\mathbf{X}, \boldsymbol{\Theta}^n}\left[\log(P(\mathbf{X}, \mathbf{z}|\boldsymbol{\Theta}))\right] \\
&= \sum_{\mathbf{z}} \log(P(\mathbf{X}, \mathbf{z}|\boldsymbol{\Theta}))P(\mathbf{z}|\mathbf{X}, \boldsymbol{\Theta}^n)
\end{aligned}
$$

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Expectation Maximization Algorithm

- It can be proven that if we choose

$$\boldsymbol{\Theta}^{n+1} = arg \max_{\boldsymbol{\Theta}} Q(\boldsymbol{\Theta}, \boldsymbol{\Theta}^{n})$$

then $L(\boldsymbol{\Theta}^{n+1}) \geq L(\boldsymbol{\Theta}^{n})$

- In many cases, finding the maximum for the Q function may be easier than likelihood function w.r.t. the parameters.

- Solution is dependent on finding a good choice of the hidden variables which eases the computation

- Iteratively improve the log-likelihood function.

EM Algorithm Summary

- Initialize with a set of model parameters (n=1)
- Compute the conditional expectation (E-step)

$$E_{\mathbf{z}|\mathbf{X},\mathbf{\Theta}^n}\left[\log(P(\mathbf{X},\mathbf{z}|\mathbf{\Theta})\right]$$

- Maximize the conditional expectation w.r.t. parameter. (M-step) (n = n+1)
- Check for convergence
- Go back to E-step if model has not converged.

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EM Algorithm for GMM

- The hidden variables $\mathbf{z}_i = l$ will be the index of the mixture component which generated $\mathbf{x}_i$
- Re-estimation formulae

$$\alpha_\ell^{new} = \frac{1}{N}\sum_{i=1}^{N} p(\ell|x_i, \Theta^g)$$

$$\mu_\ell^{new} = \frac{\sum_{i=1}^{N} x_i p(\ell|x_i, \Theta^g)}{\sum_{i=1}^{N} p(\ell|x_i, \Theta^g)}$$

$$\Sigma_\ell^{new} = \frac{\sum_{i=1}^{N} p(\ell|x_i, \Theta^g)(x_i - \mu_\ell^{new})(x_i - \mu_\ell^{new})^T}{\sum_{i=1}^{N} p(\ell|x_i, \Theta^g)}$$

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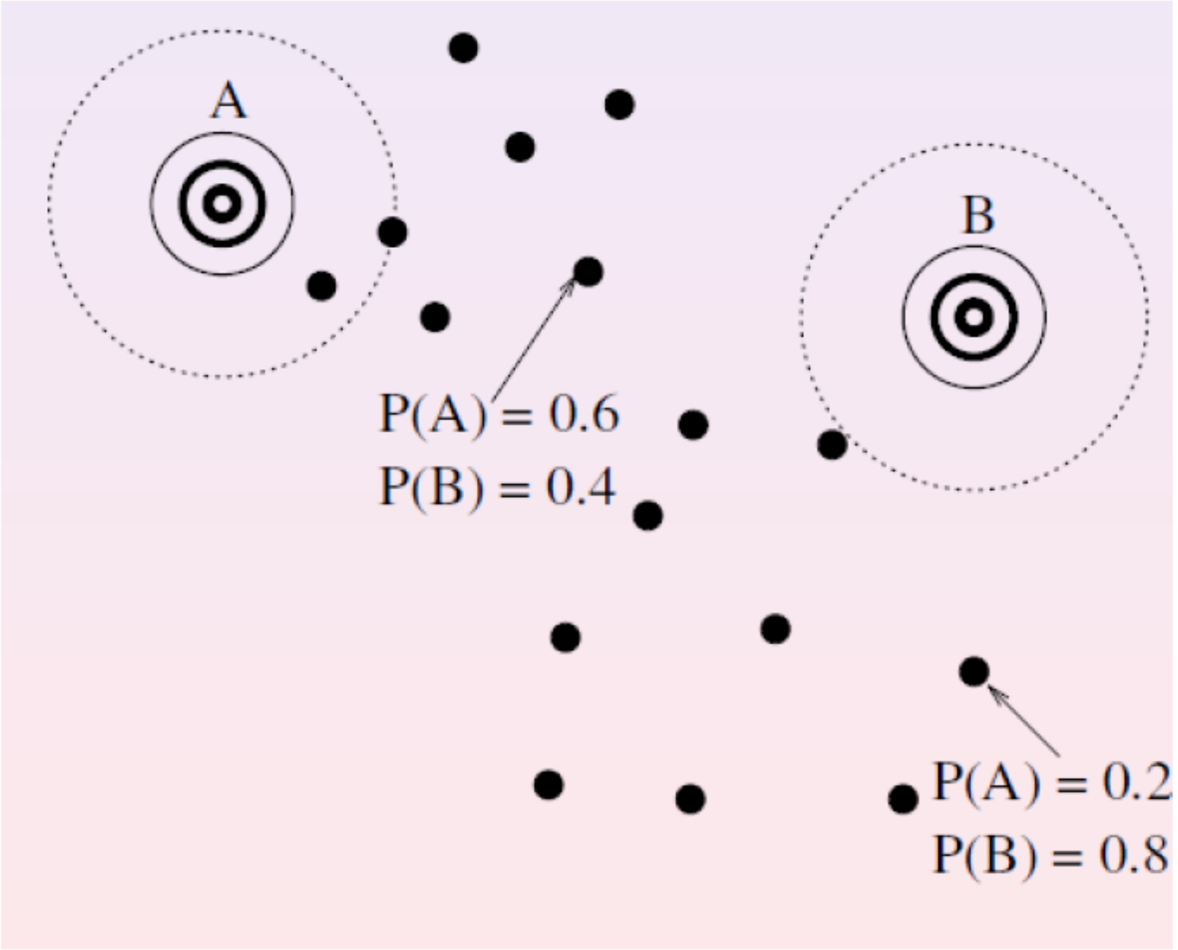
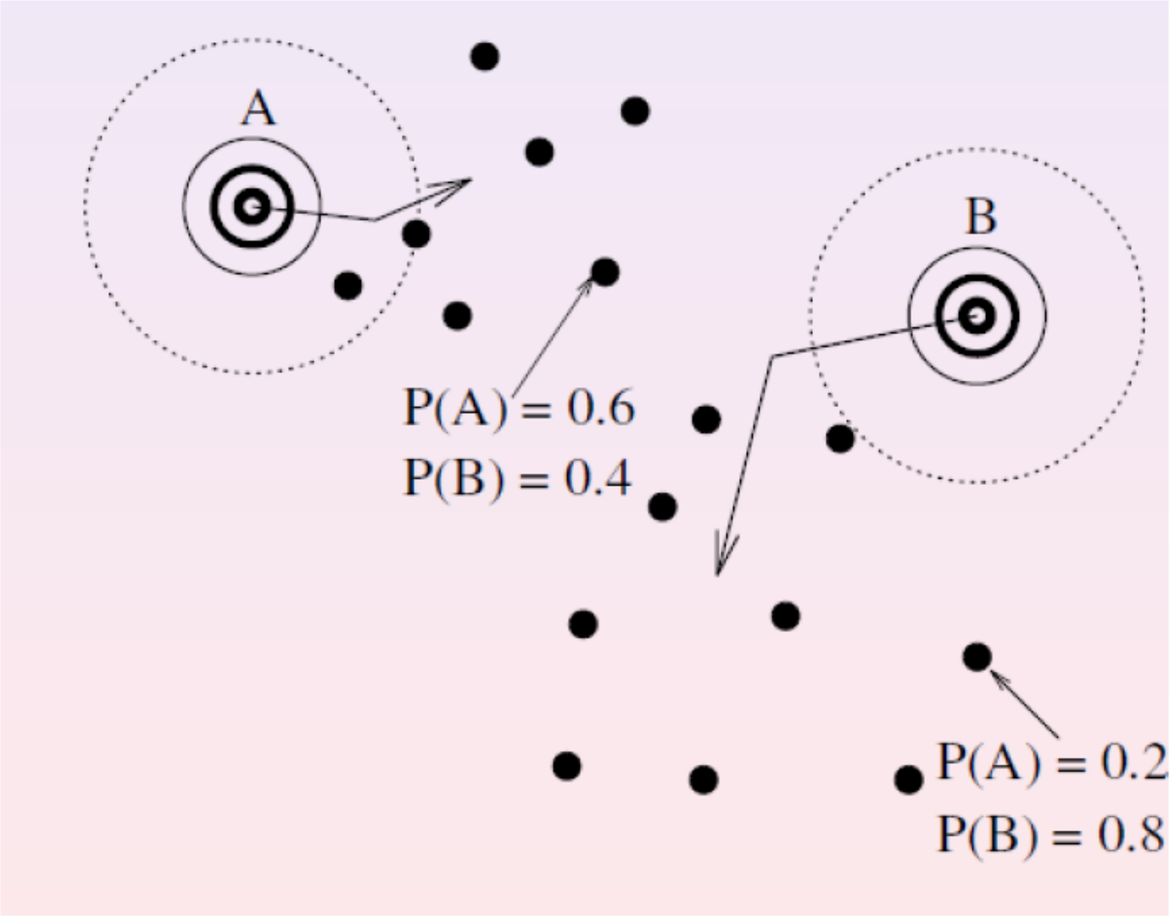
EM Algorithm for GMM

E-step



M-step



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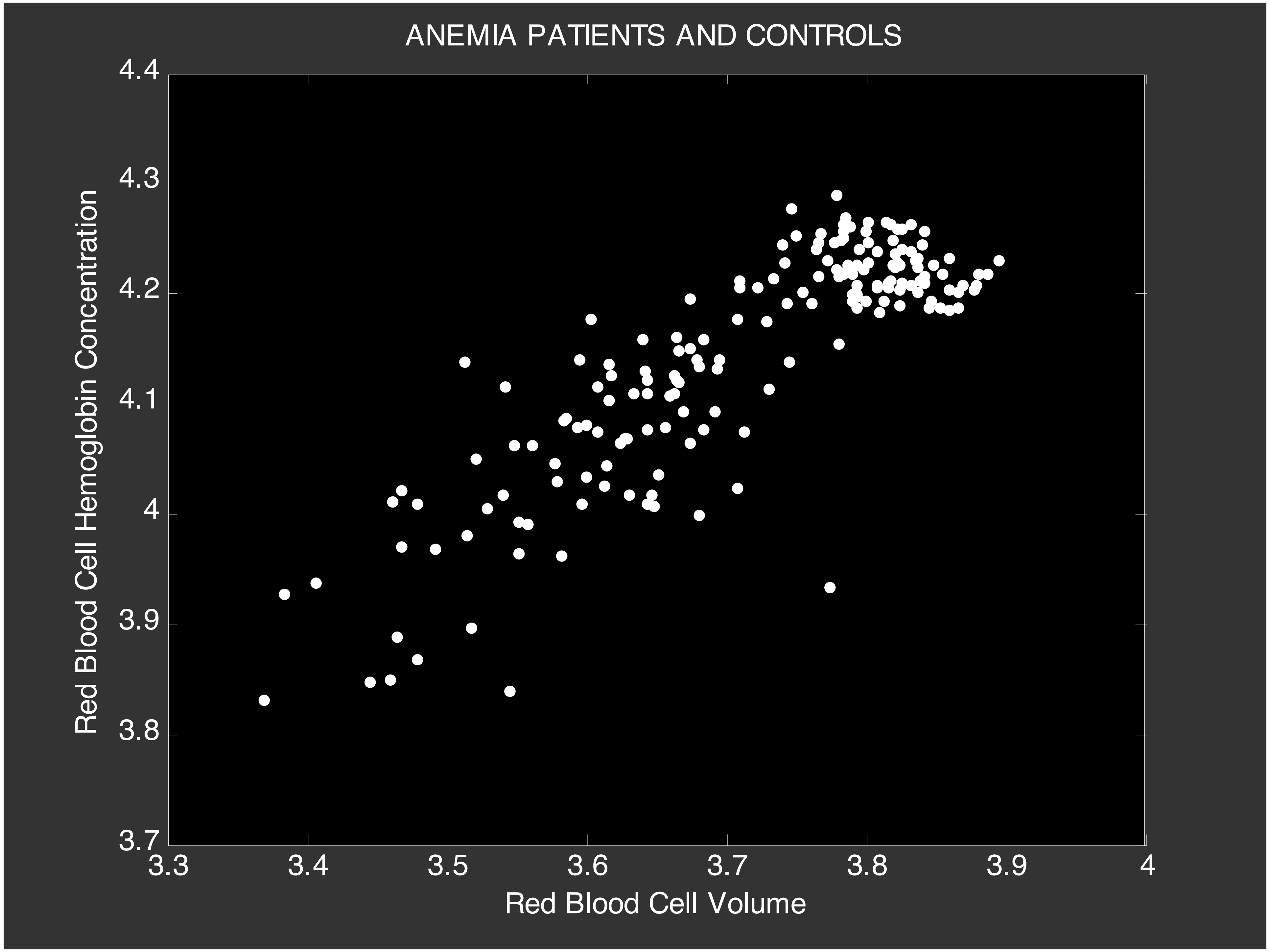

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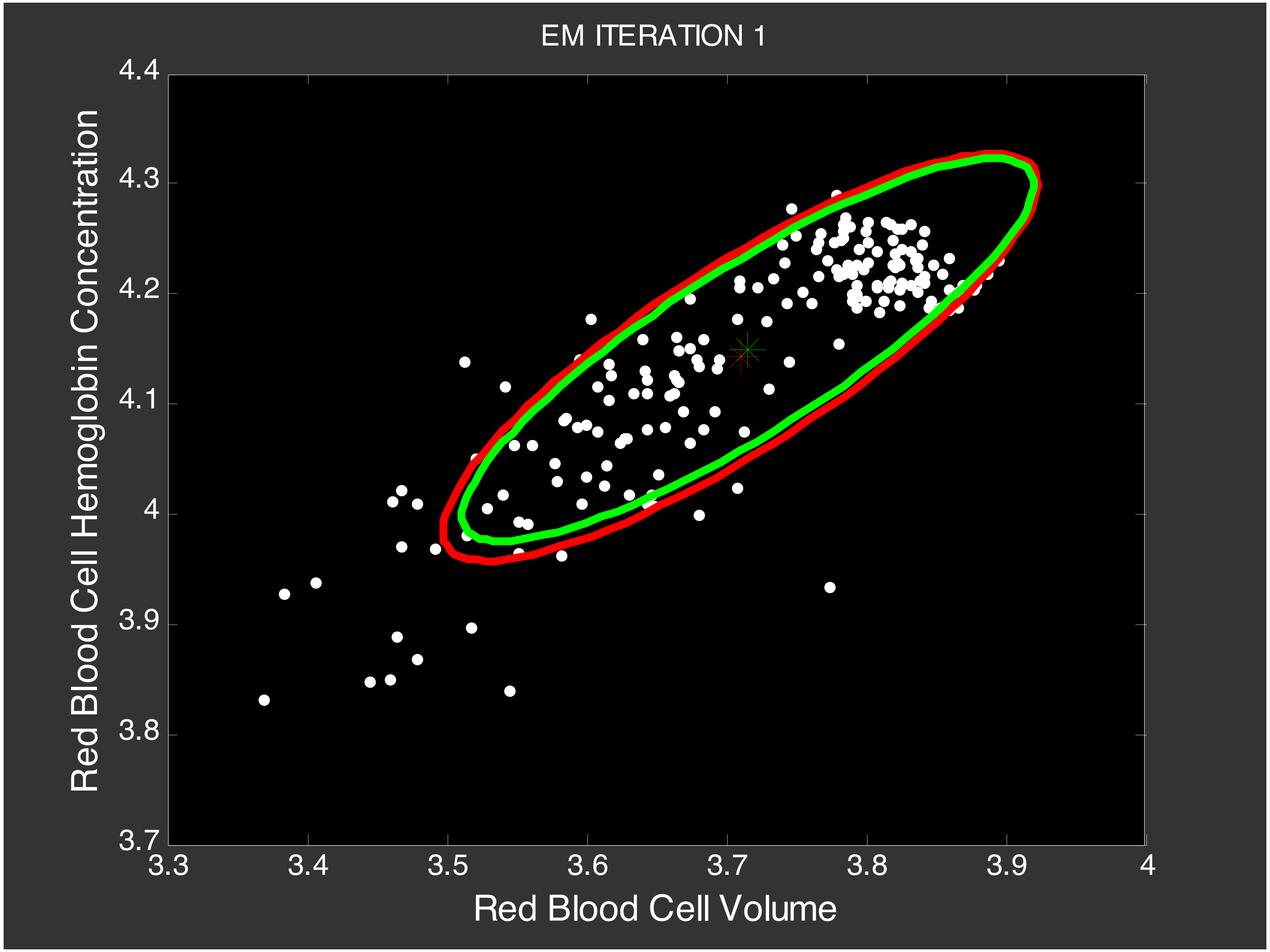

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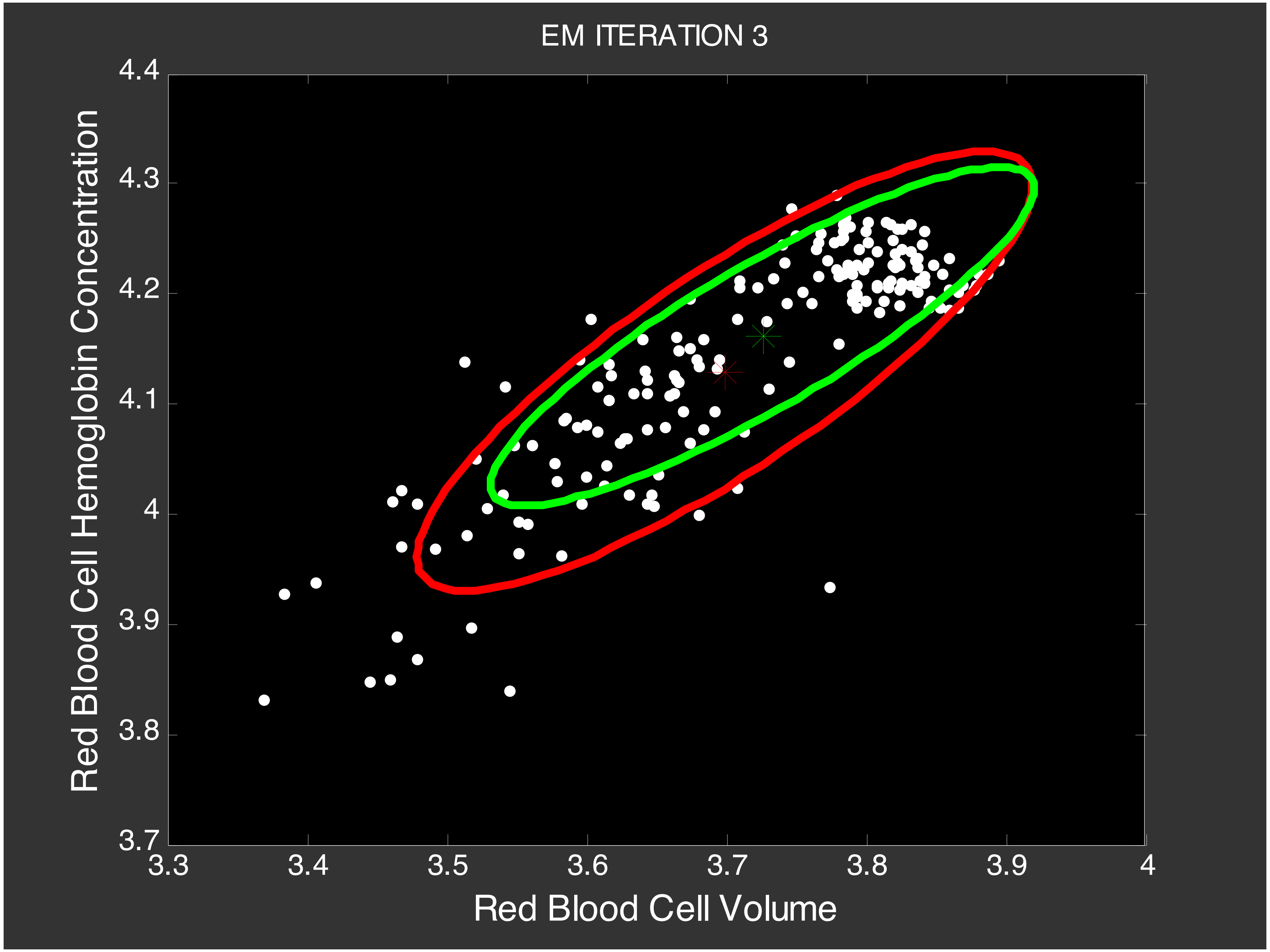

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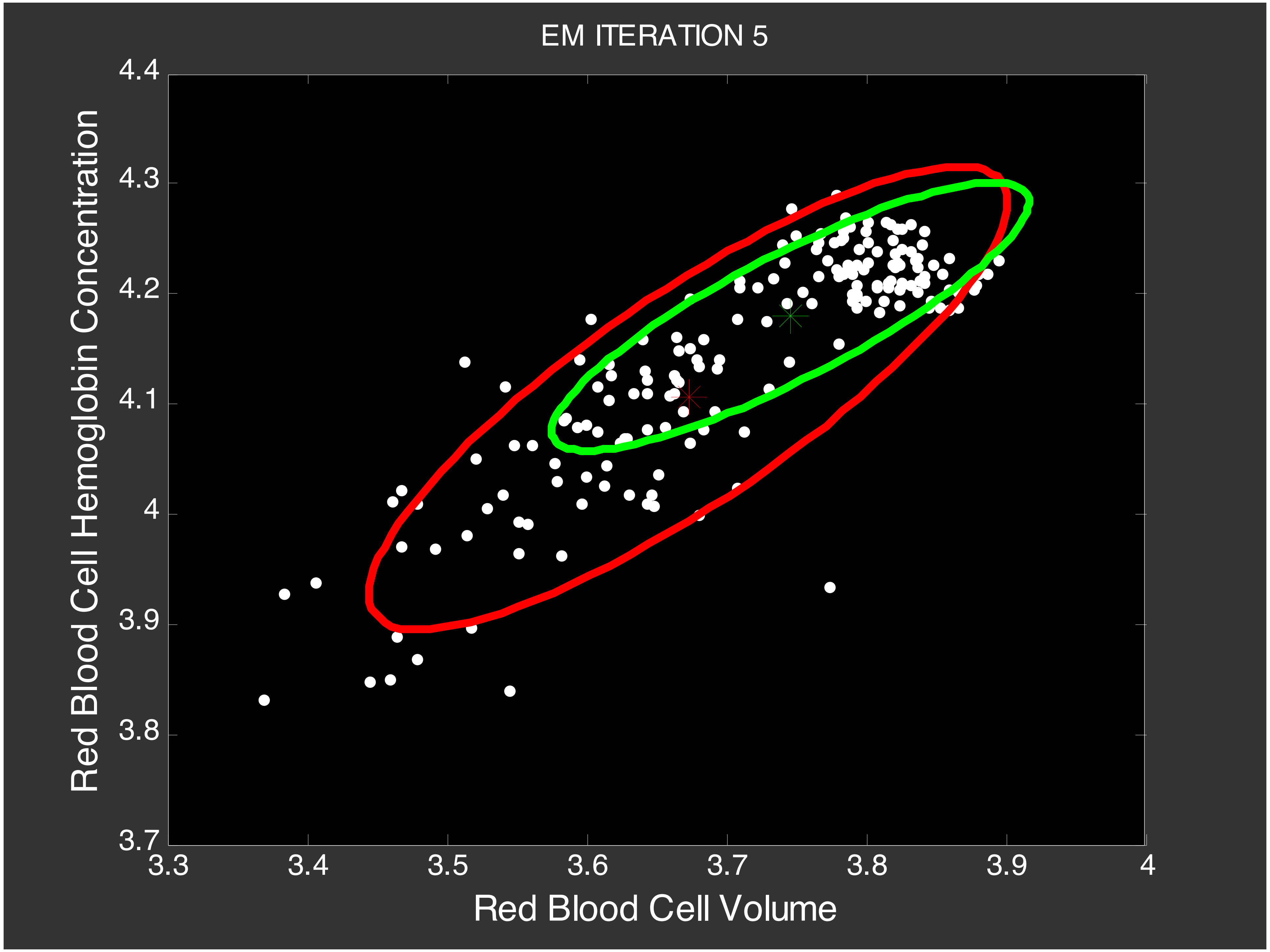

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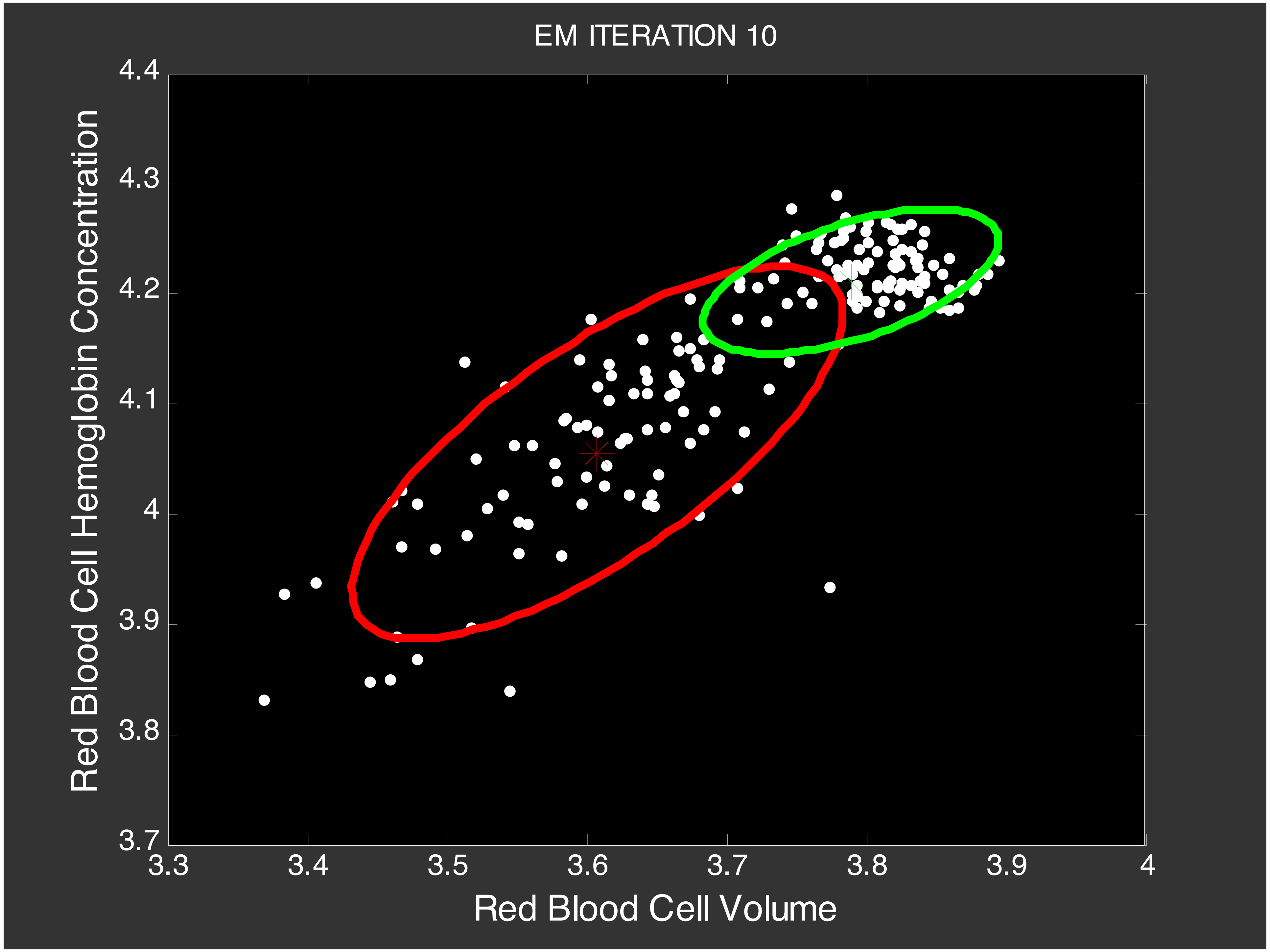
EM Algorithm for GMM



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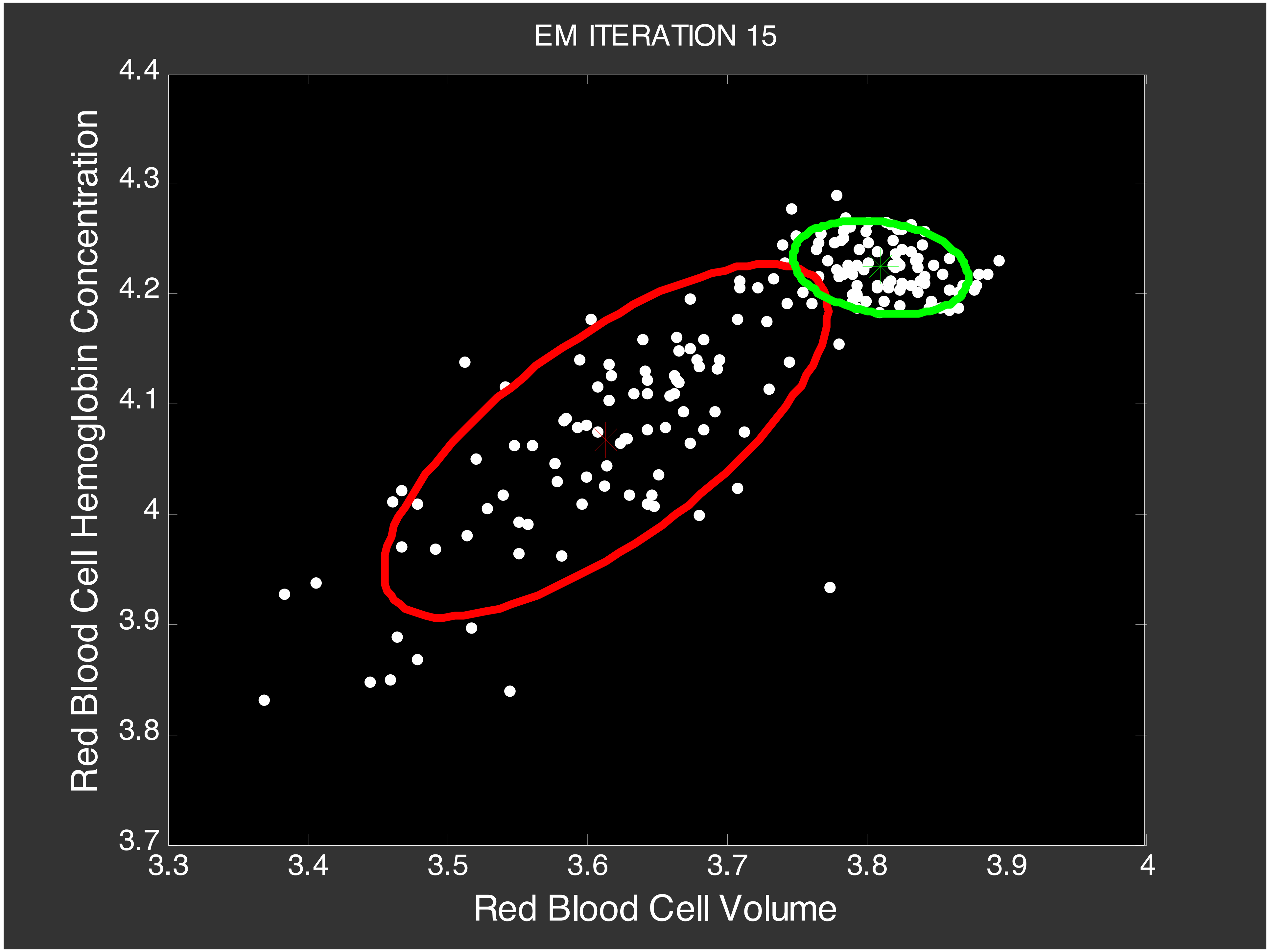

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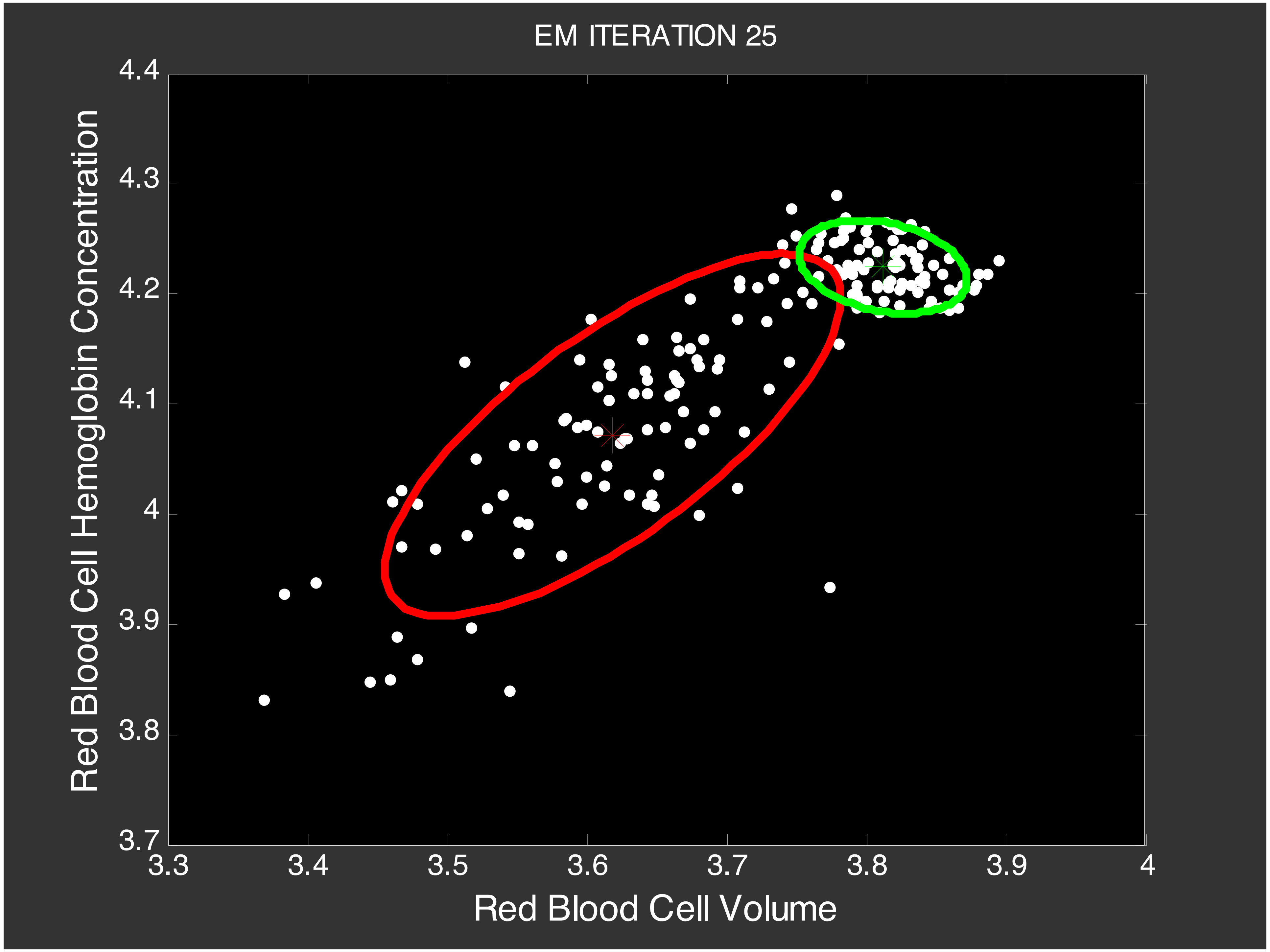

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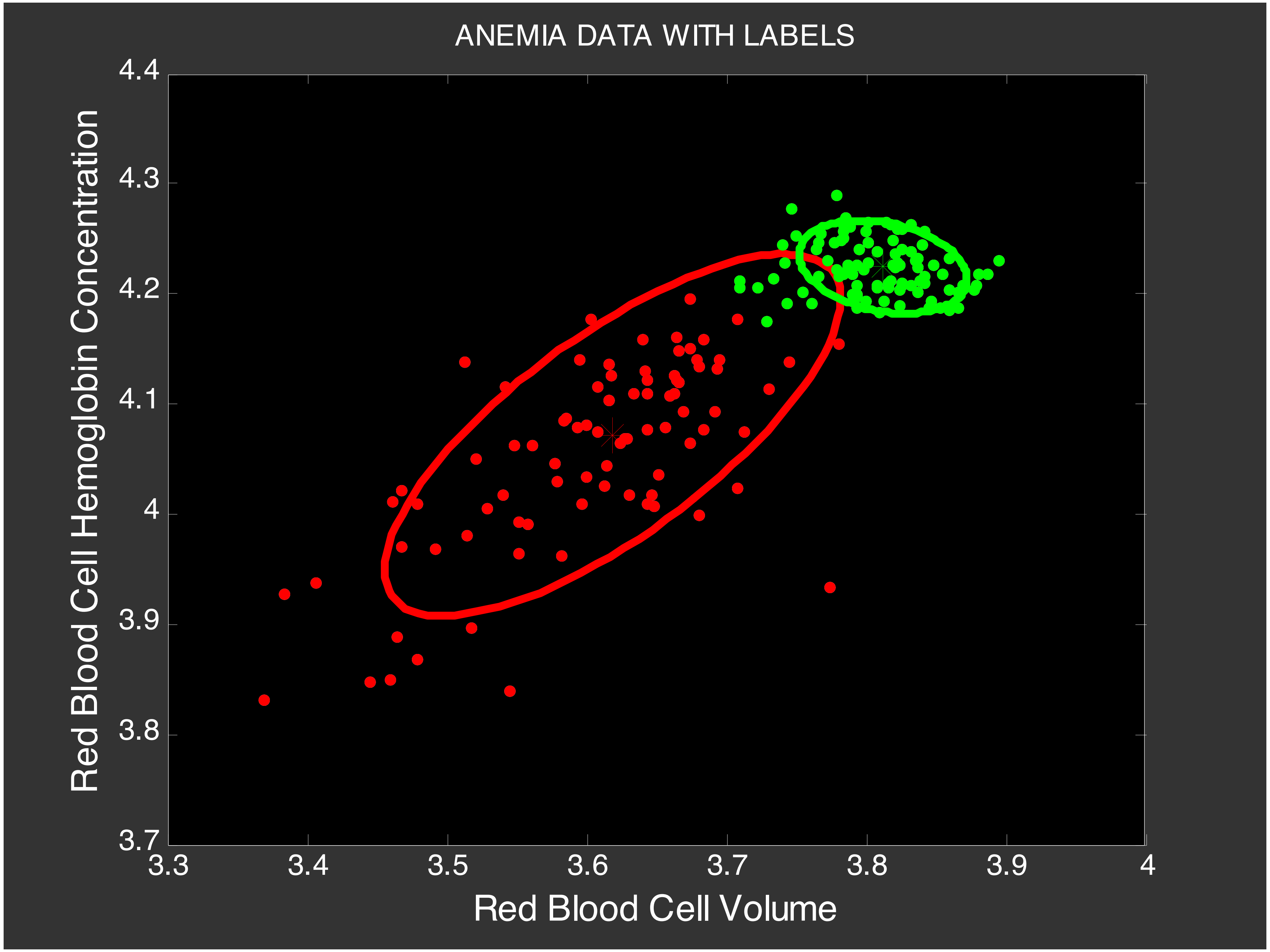
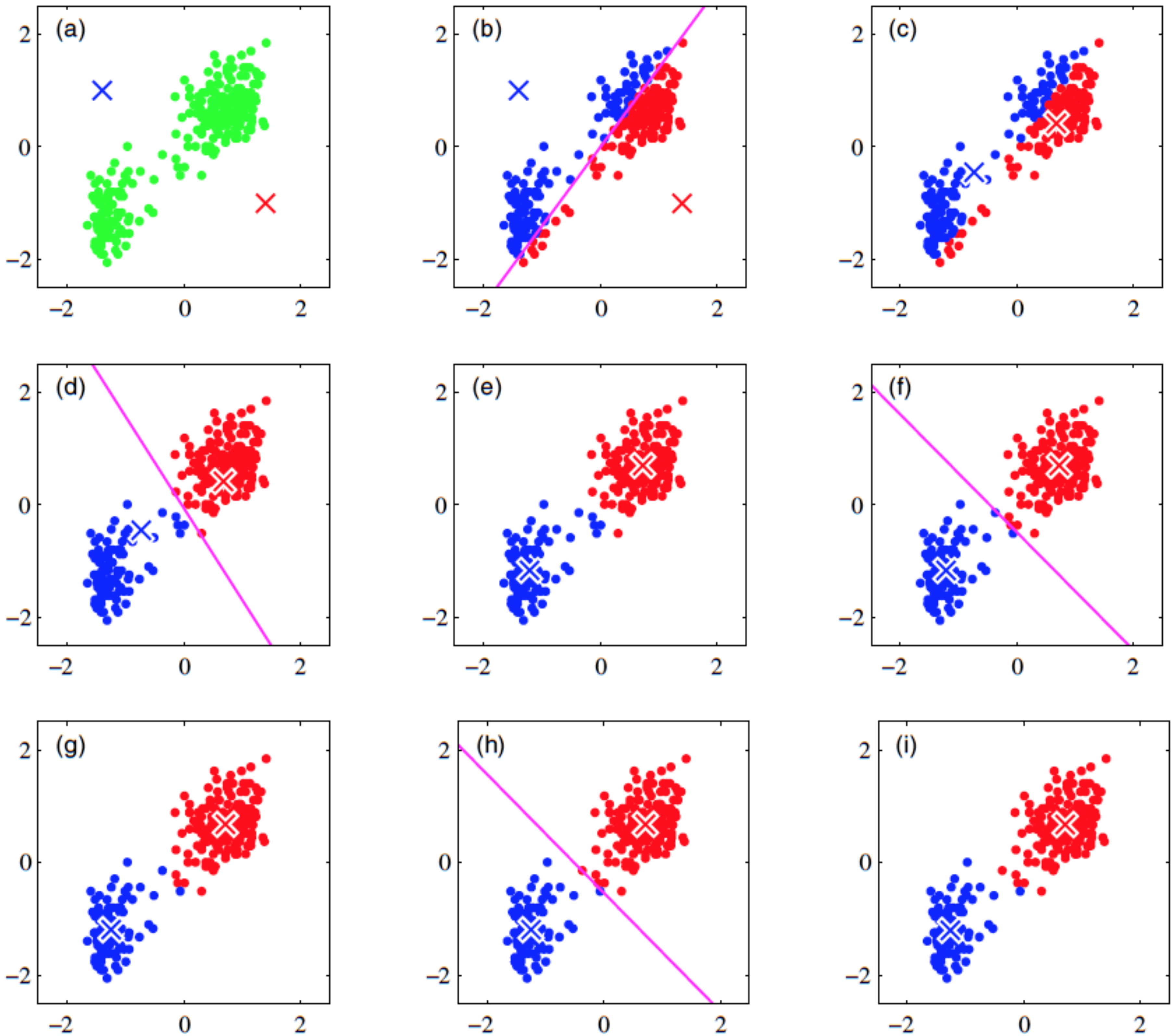
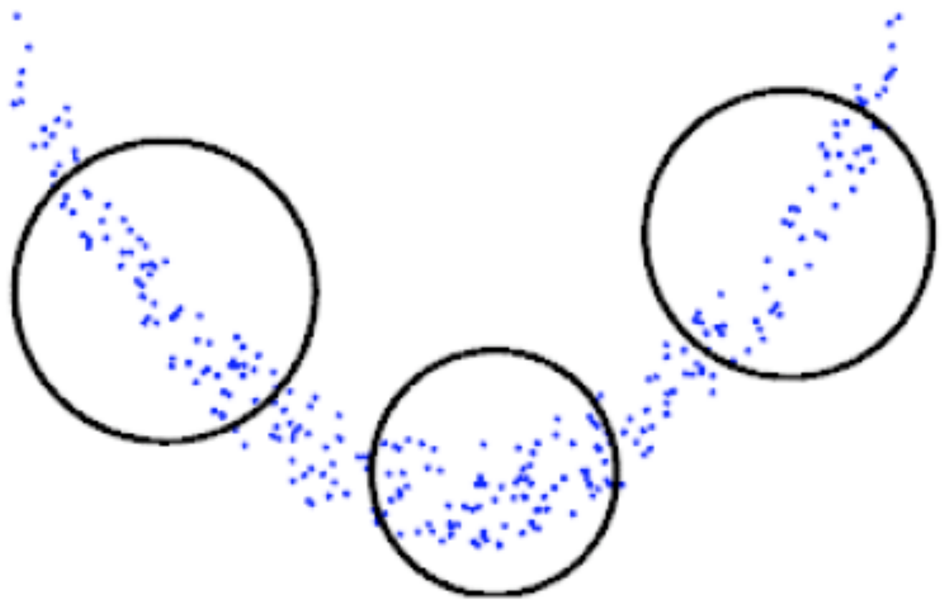


K-means Algorithm for Initialization



Other Considerations

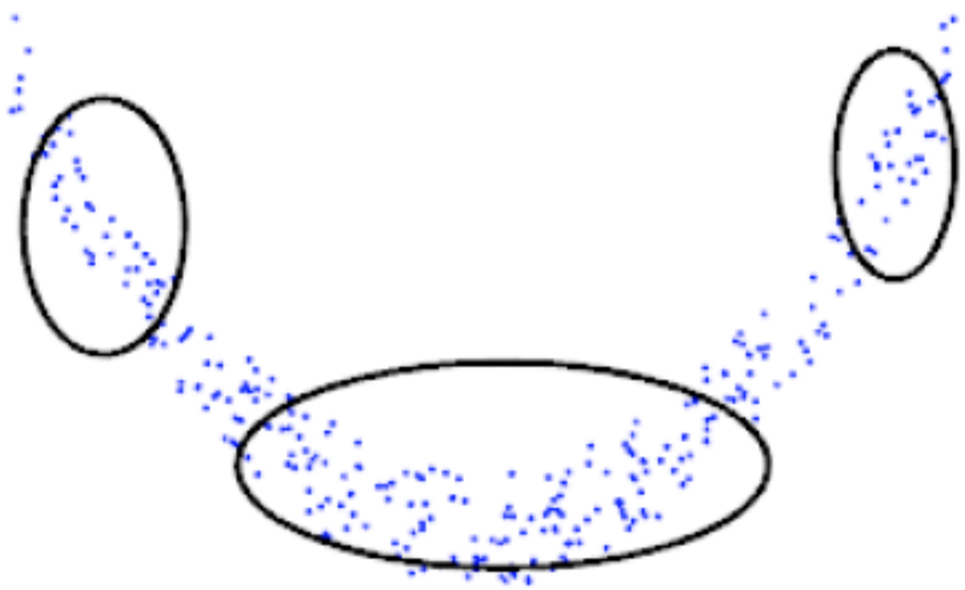
- Initialization - random or k-means
- Number of Gaussians
- Type of Covariance matrix
  - Spherical covariance



-Less precise.
-Very efficient to compute.

Other Considerations

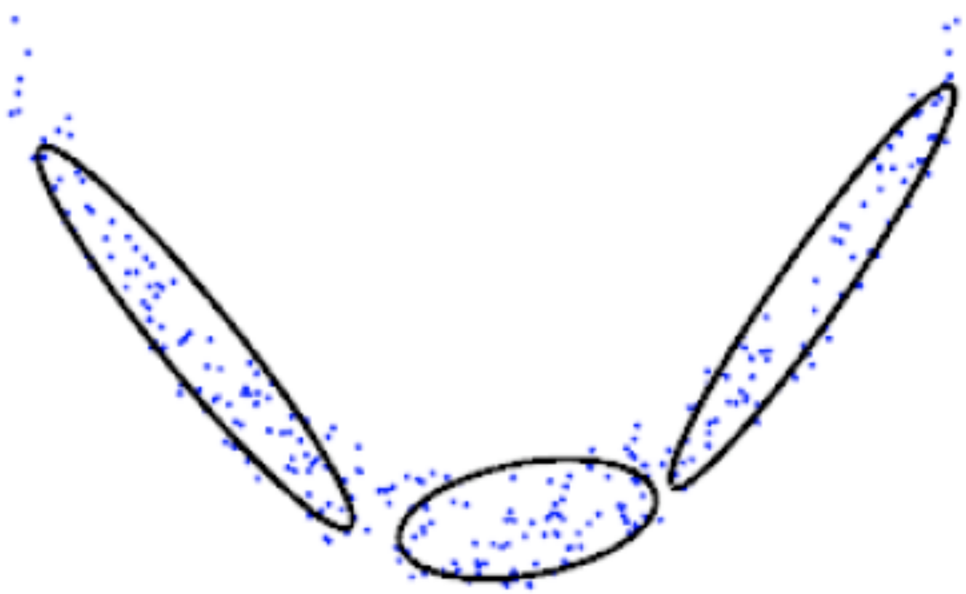
- Initialization - random or k-means
- Number of Gaussians
- Type of Covariance matrix
    - Diagonal covariance



-More precise.
-Efficient to compute.

Other Considerations

- Initialization - random or k-means
- Number of Gaussians
- Type of Covariance matrix
    - Full covariance



-Very precise.
-Less efficient to compute.