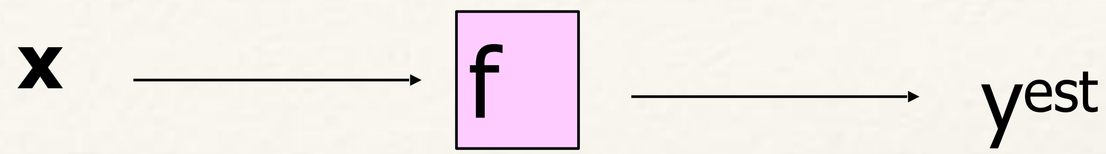
E9 205 Machine Learning for Signal Procesing

Support Vector Machines

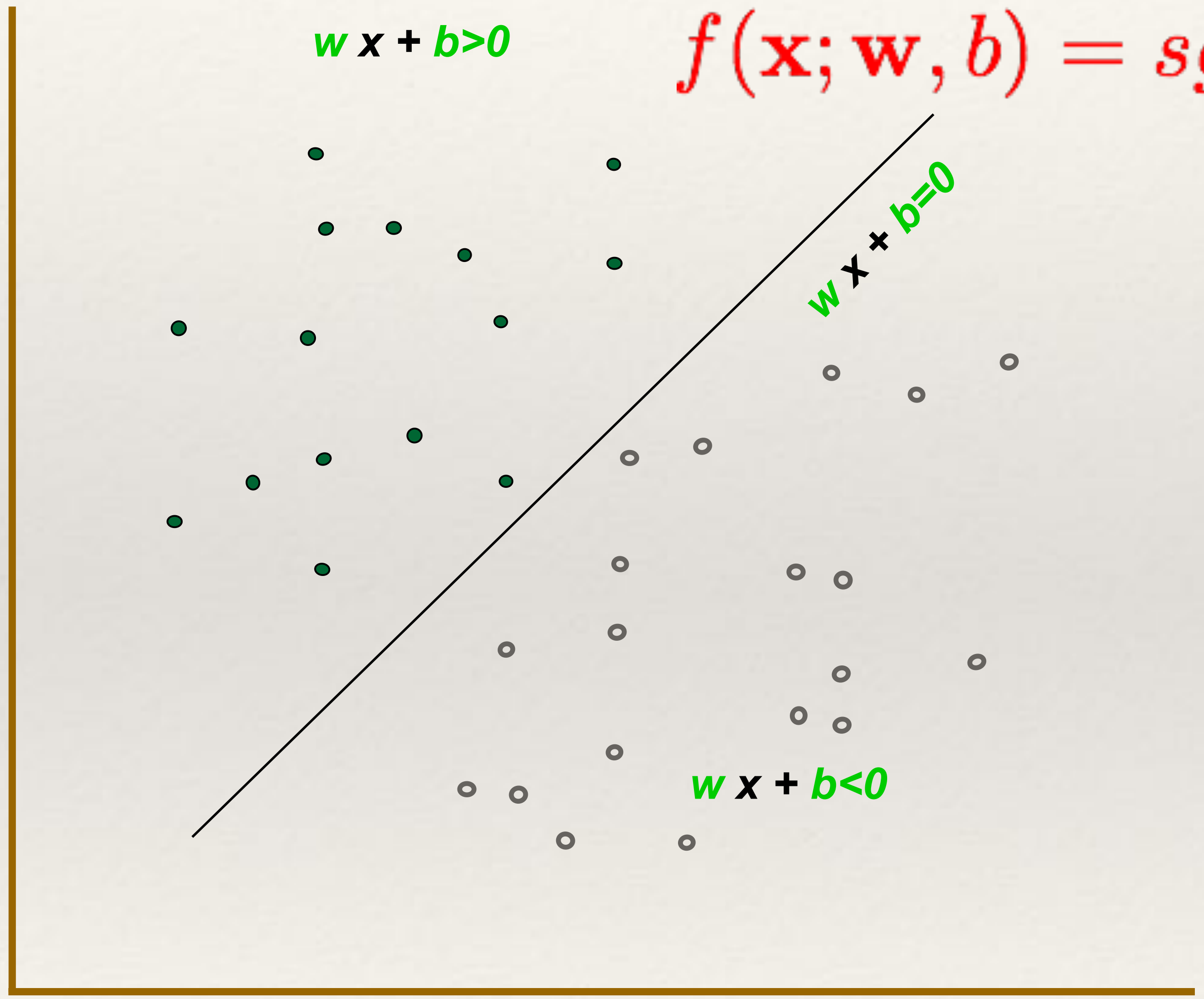
27-10-2017

Linear Classifiers



$$f(\mathbf{x}; \mathbf{w}, b) = sgn(\mathbf{w}^T \boldsymbol{\phi}(\mathbf{x}) + b)$$

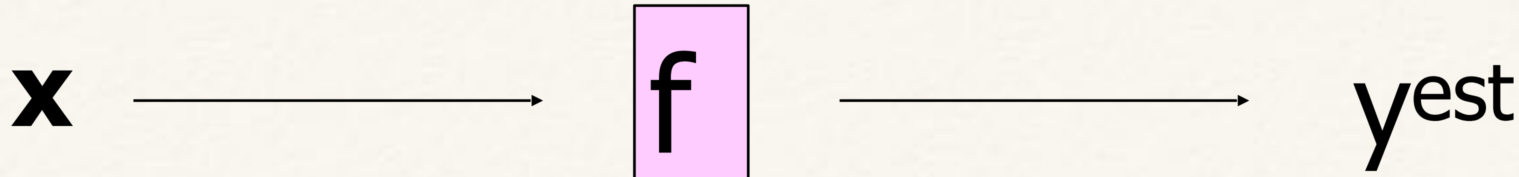
- denotes +1
- denotes -1



How would you classify this data?

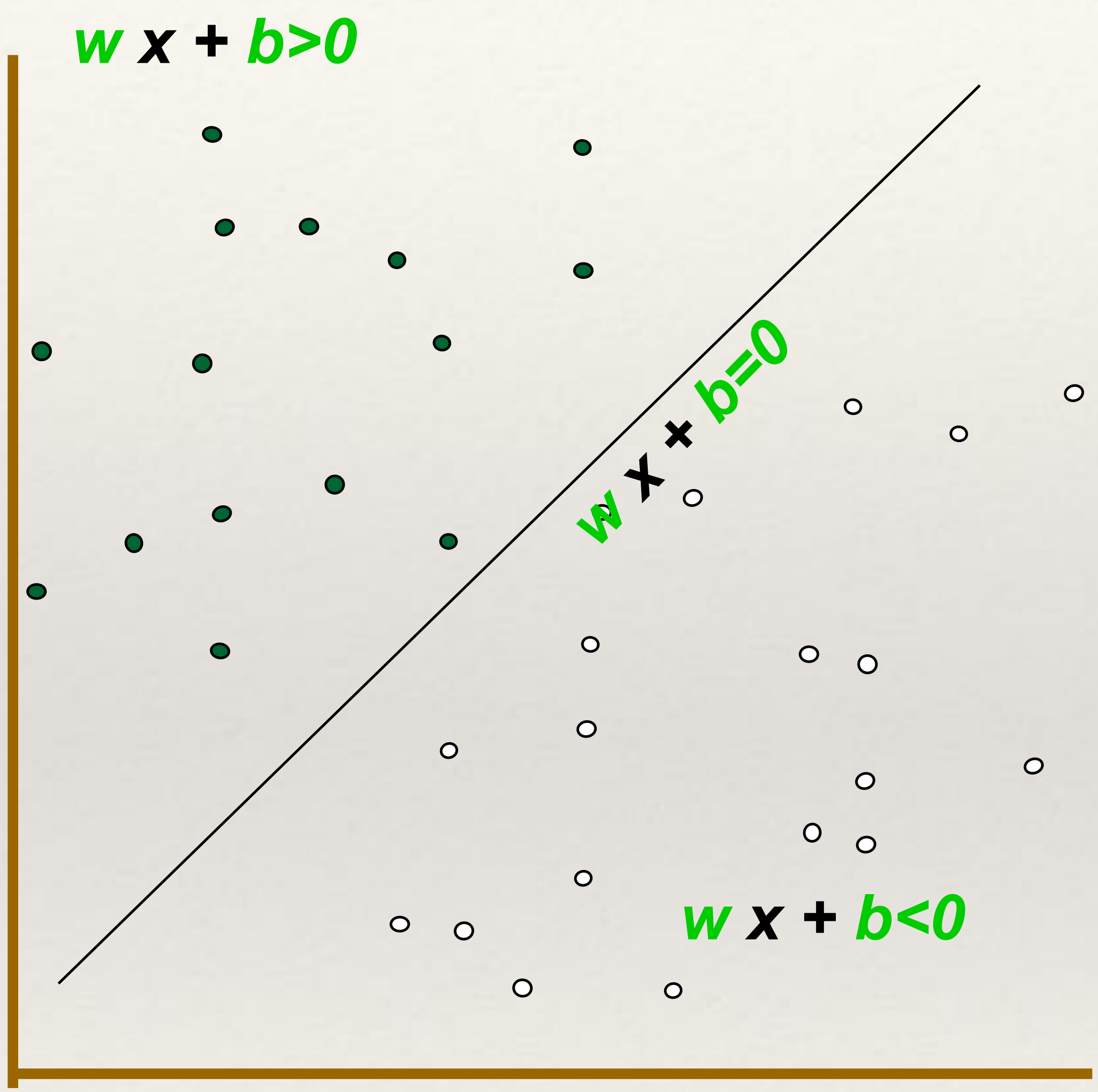
"SVM and applications", Mingyue Tan. Univ of British Columbia

Linear Classifiers



$$f(\mathbf{x}; \mathbf{w}, b) = sgn(\mathbf{w}^T \phi(\mathbf{x}) + b)$$

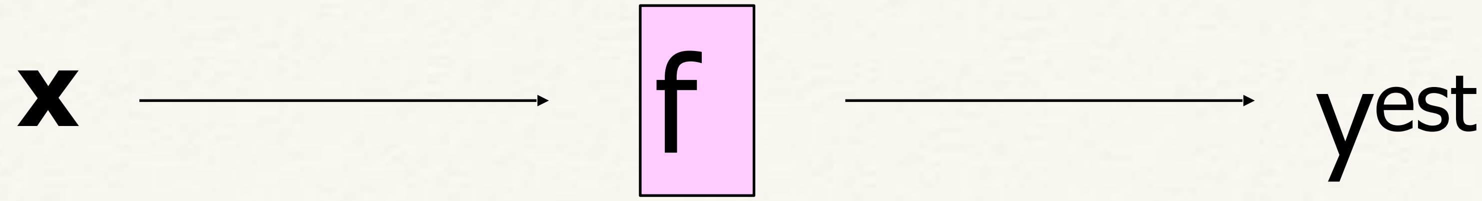
- denotes +1
- ◦ denotes -1



How would you classify this data?

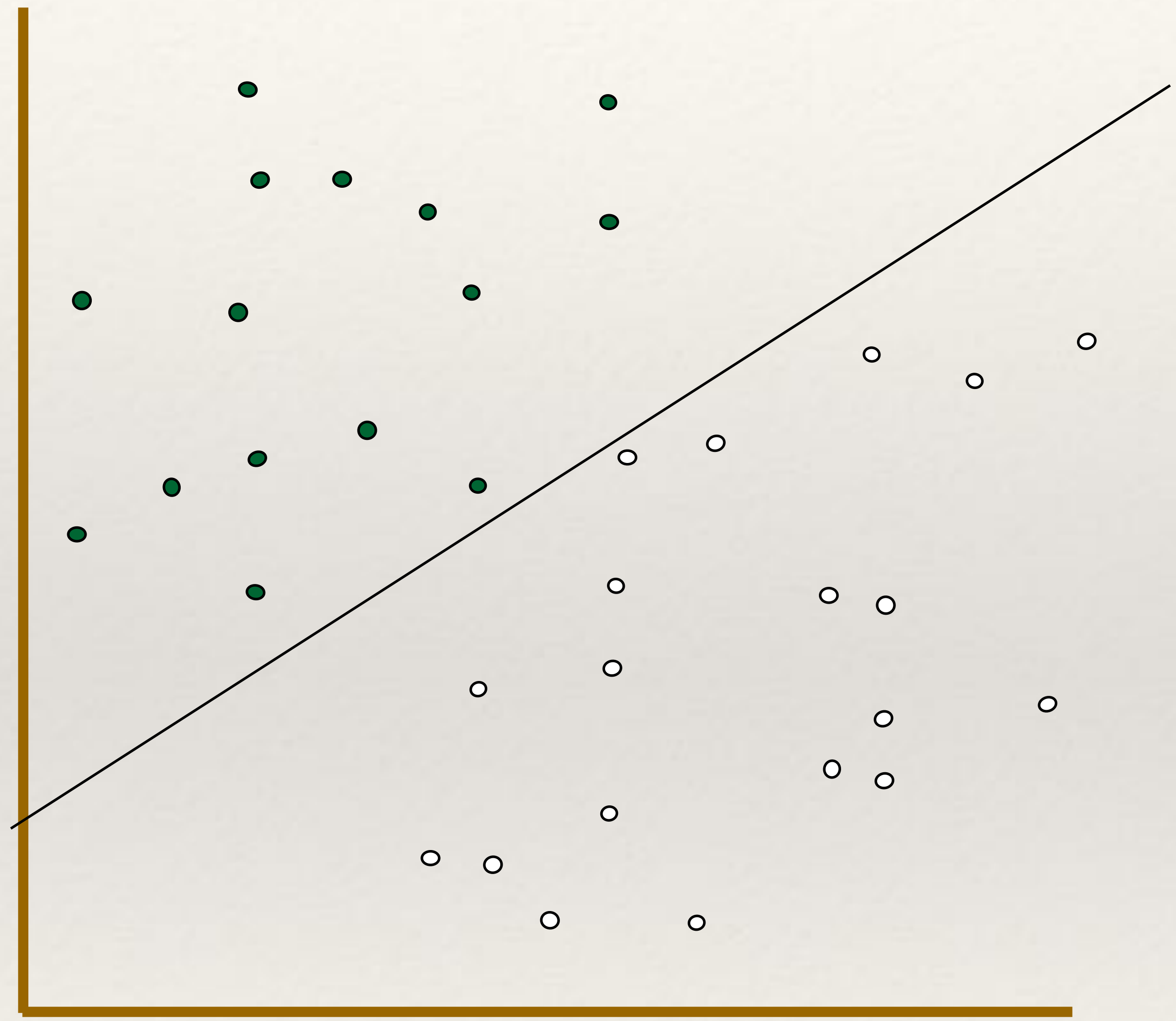
"SVM and applications", Mingyue Tan. Univ of British Columbia

Linear Classifiers



$$f(\mathbf{x};\mathbf{w},b) = sgn(\mathbf{w}^T\boldsymbol{\phi}(\mathbf{x}) + b)$$

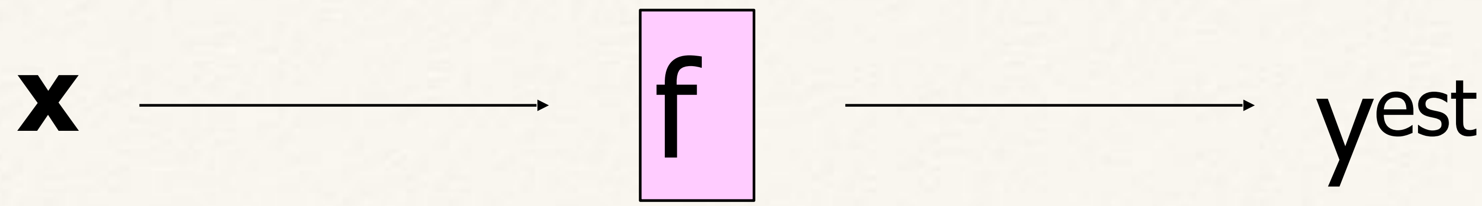
- • denotes +1
- ◦ denotes -1



How would you classify this data?

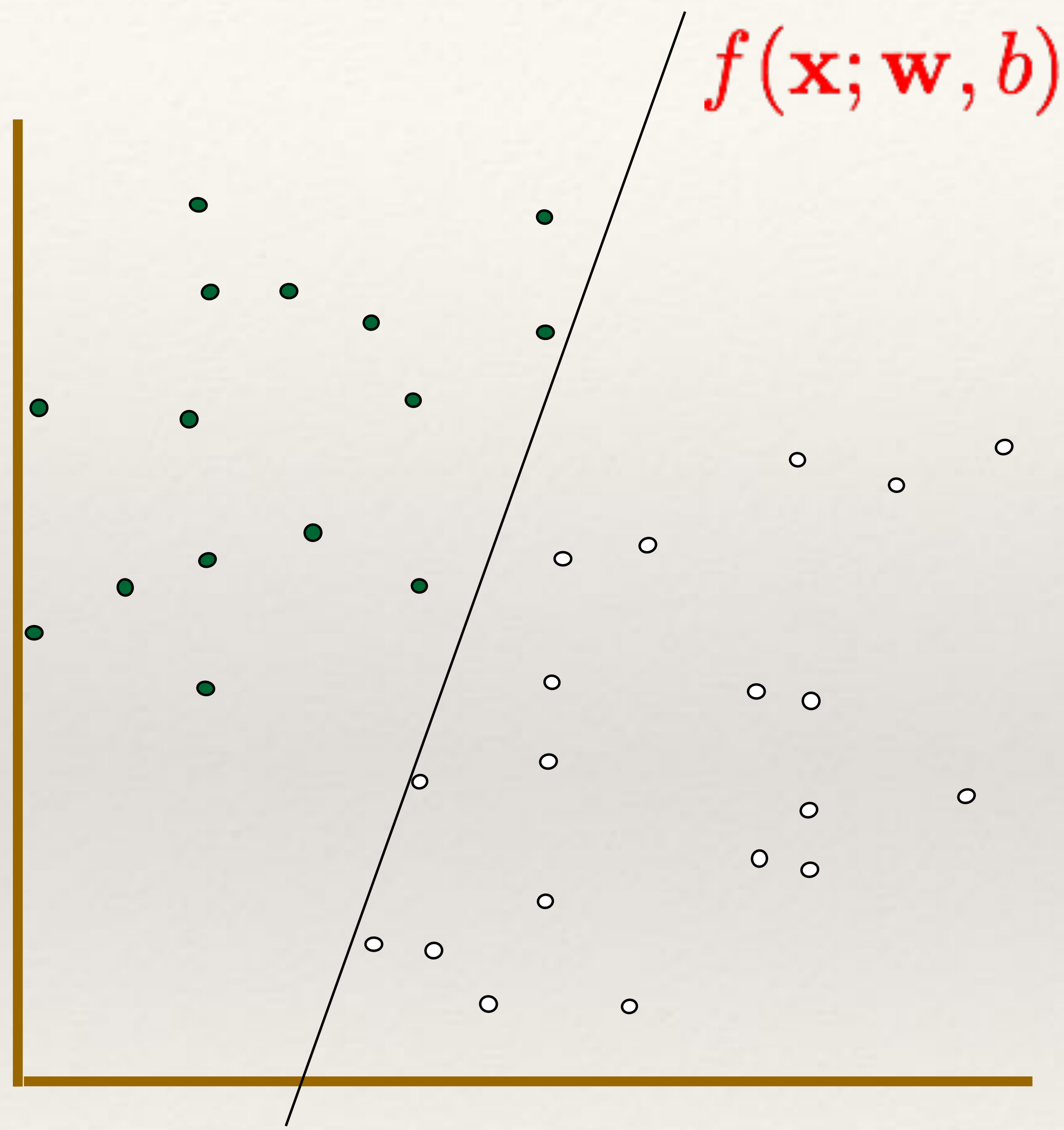
“SVM and applications”, Mingyue Tan. Univ of British Columbia

Linear Classifiers



$$f(\mathbf{x}; \mathbf{w}, b) = sgn(\mathbf{w}^T \boldsymbol{\phi}(\mathbf{x}) + b)$$

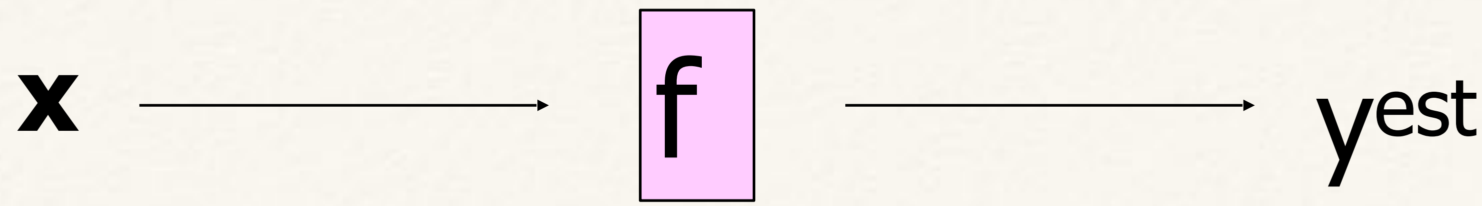
- • denotes +1
- ◦ denotes -1

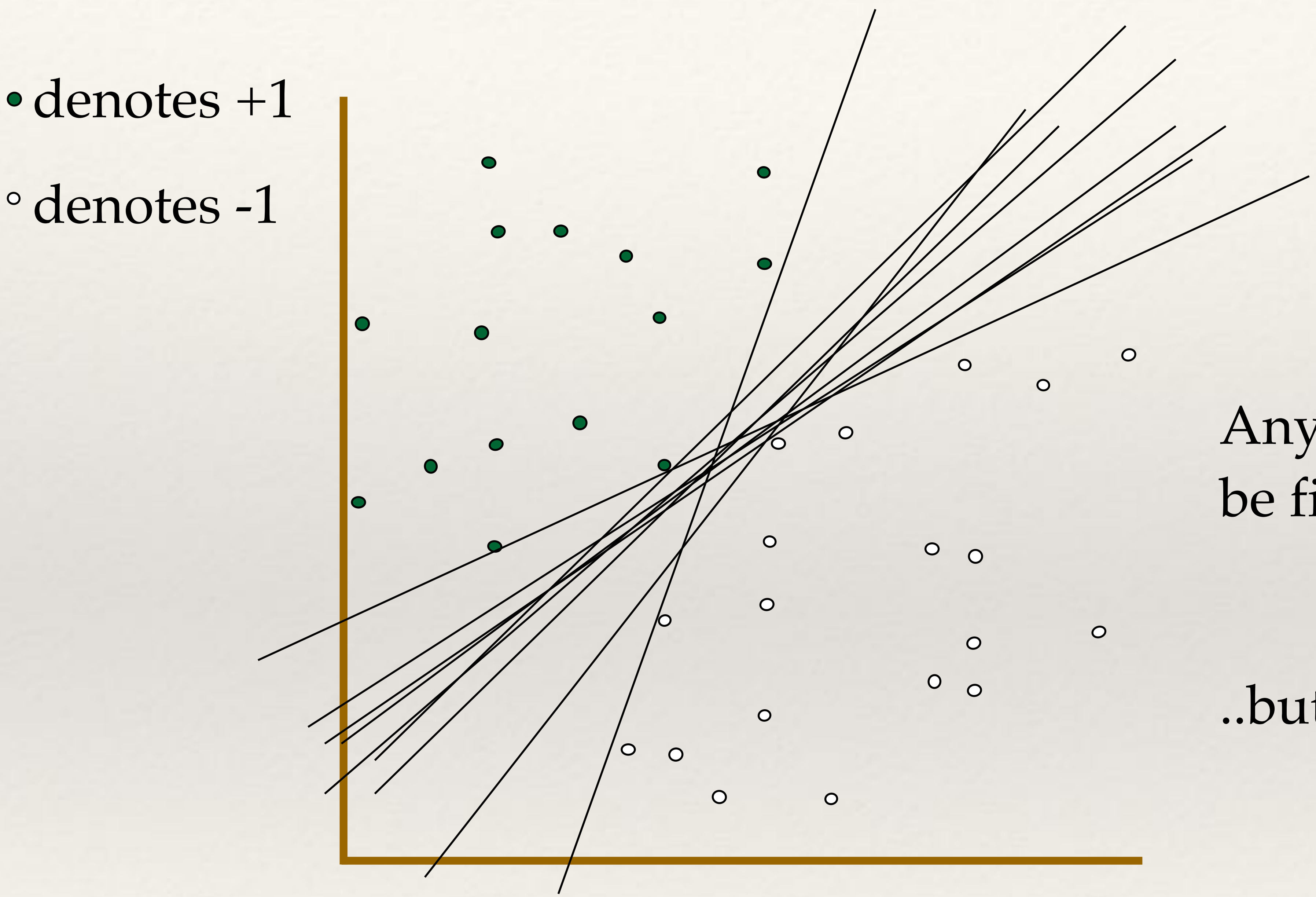


How would you classify this data?

"SVM and applications", Mingyue Tan. Univ of British Columbia

Linear Classifiers

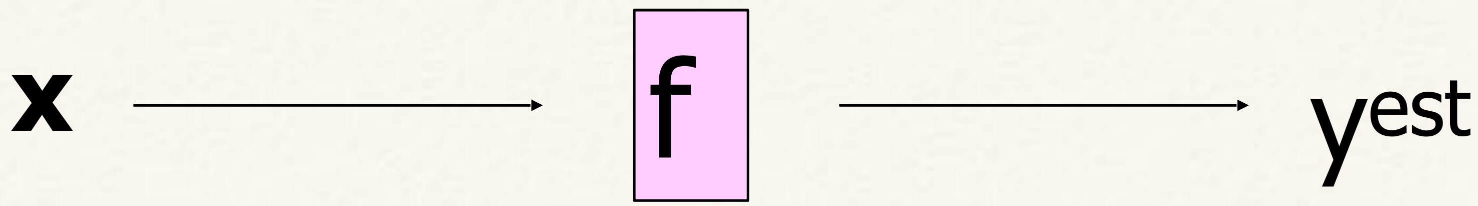




Any of these would be fine..

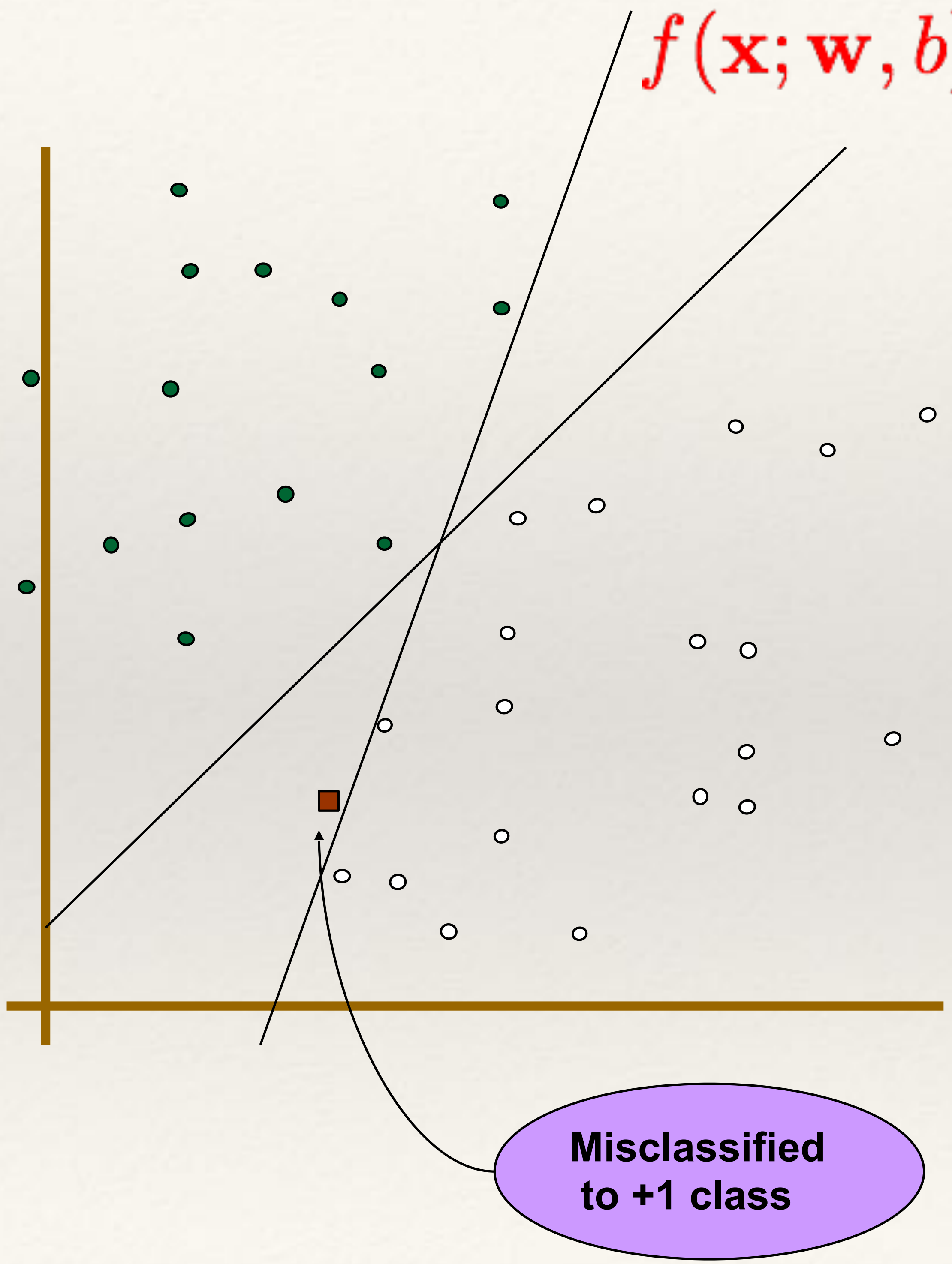
..but which is best?

Linear Classifiers



$$f(\mathbf{x}; \mathbf{w}, b) = sgn(\mathbf{w}^T \phi(\mathbf{x}) + b)$$

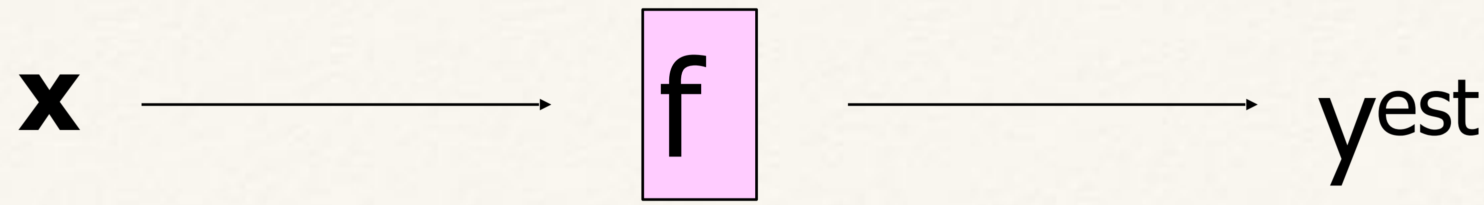
- denotes +1
- denotes -1



How would you classify this data?

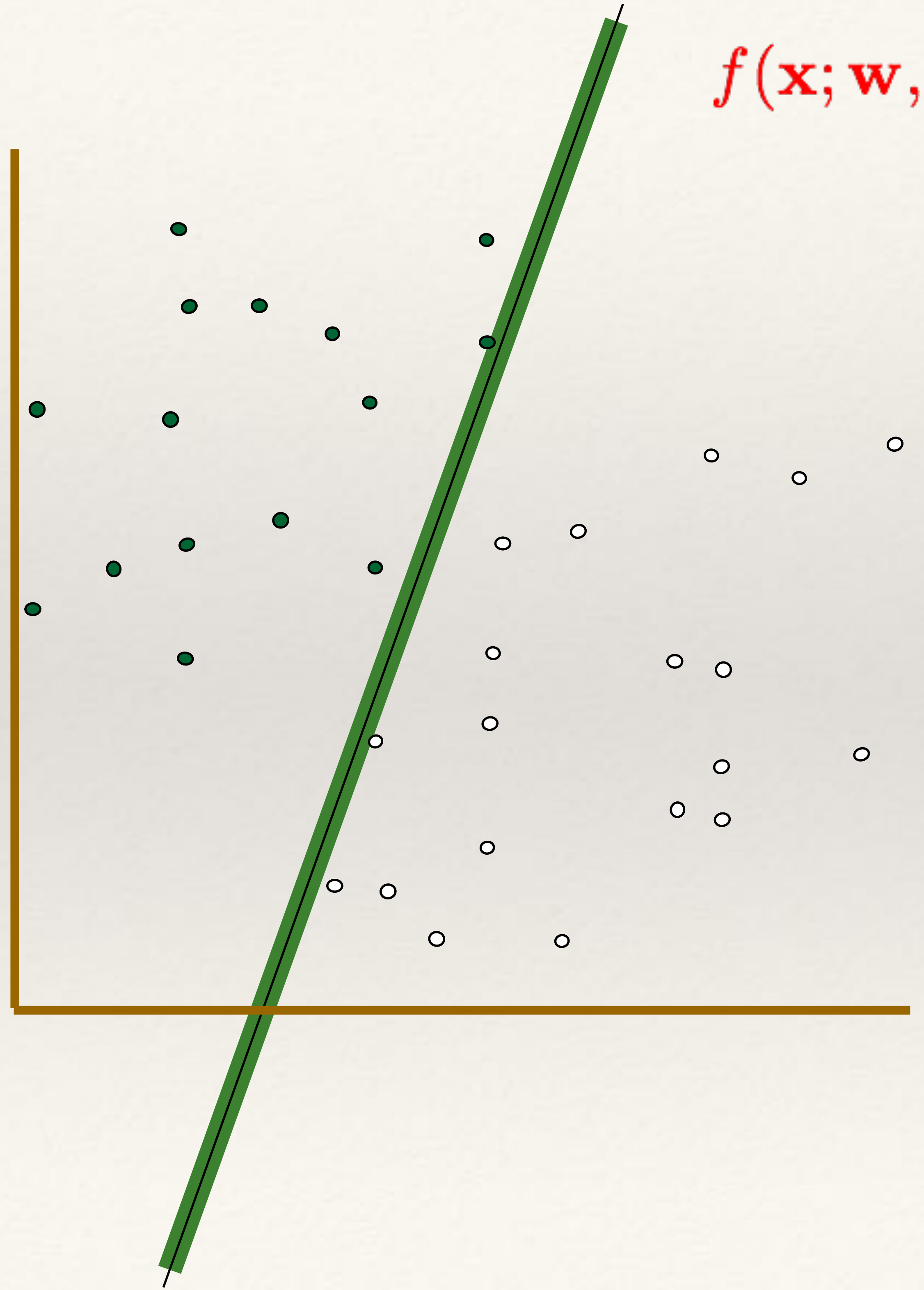
"SVM and applications", Mingyue Tan. Univ of British Columbia

Linear Classifiers



$$f(\mathbf{x};\mathbf{w},b) = sgn(\mathbf{w}^T\phi(\mathbf{x})+b)$$

- denotes +1
- ◦ denotes -1



Define the **margin** of a linear classifier as the width that the boundary could be increased by before hitting a datapoint.

Maximum Margin

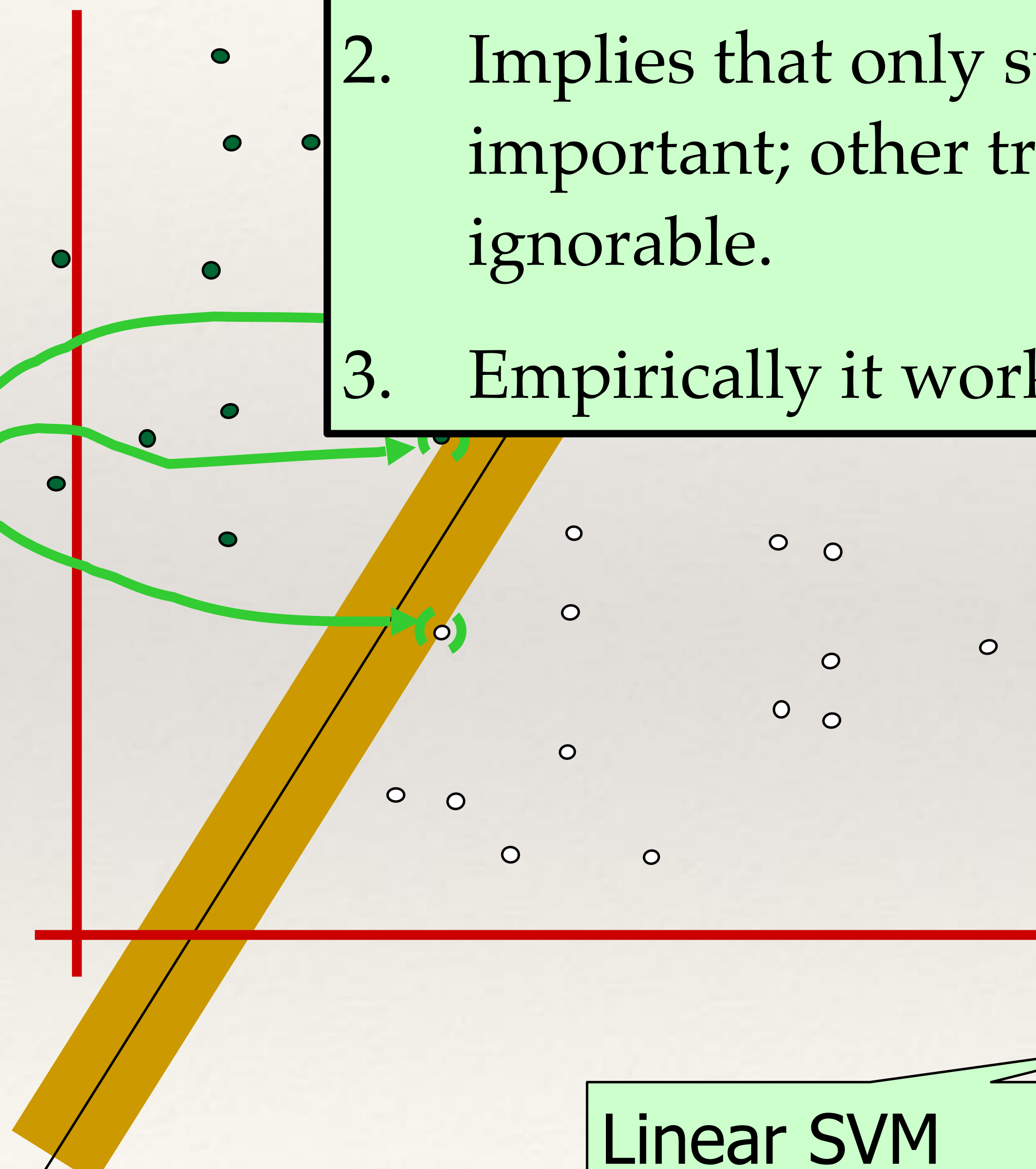
x → f → yest

1. Maximizing the margin is good according to intuition
2. Implies that only support vectors are important; other training examples are ignorable.
3. Empirically it works very very well.

• denotes +1

° denotes -1

Support Vectors are those data points that the margin pushes up against



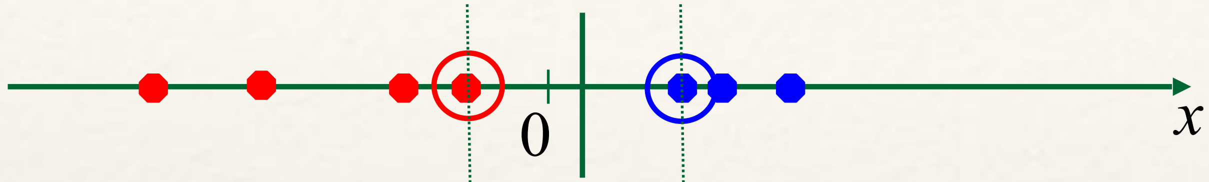
with the, um, maximum margin.

This is the simplest kind of SVM (Called an LSVM)

Linear SVM

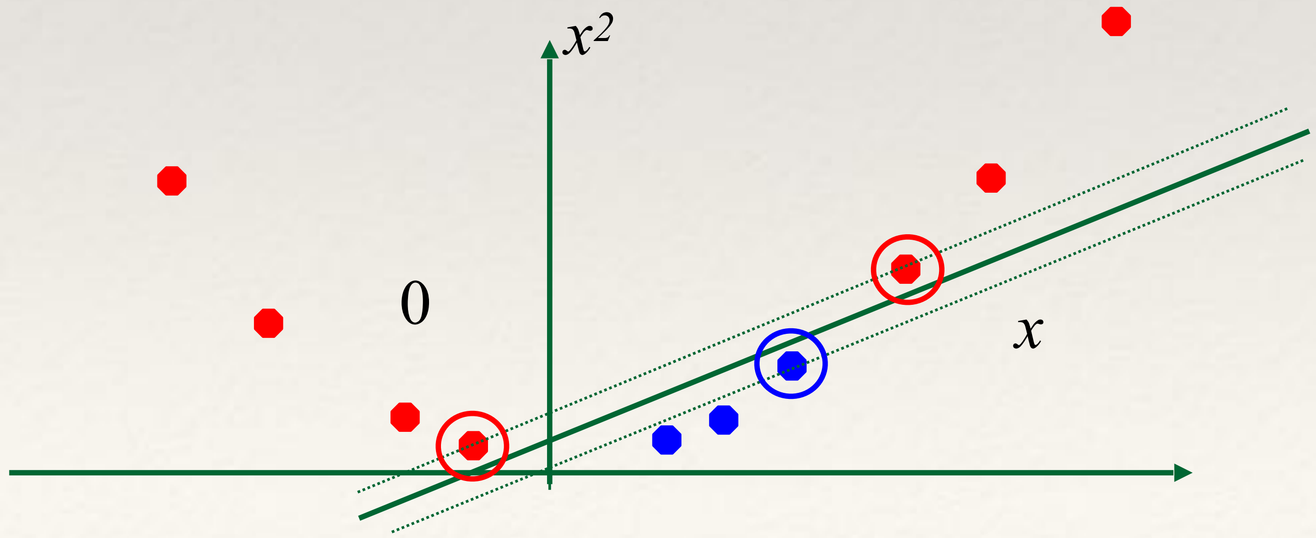
Non-linear SVMs

- Datasets that are linearly separable with some noise work out great:



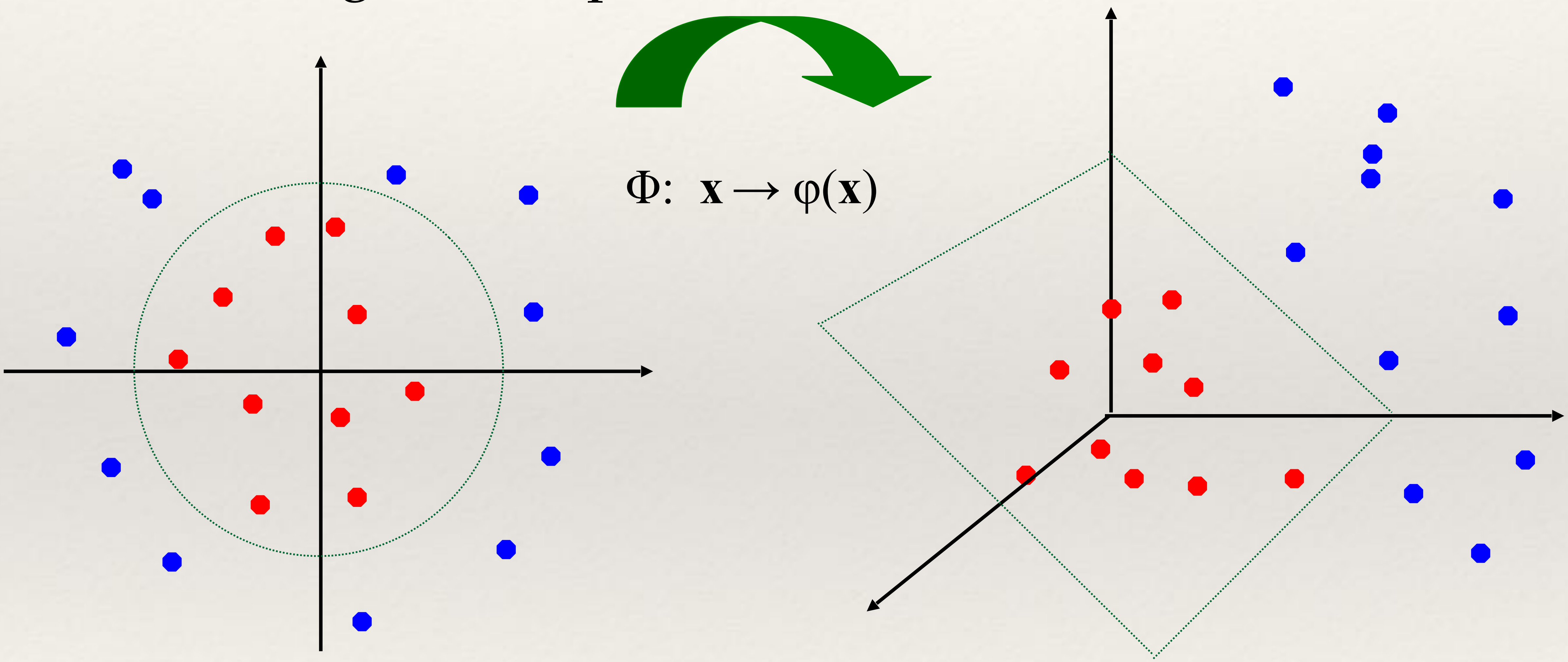
- But what are we going to do if the dataset is just too hard?

- How about… mapping data to a higher-dimensional space:



Non-linear SVMs: Feature spaces

- General idea: the original input space can always be mapped to some higher-dimensional feature space where the training set is separable:



The "Kernel Trick"

- The linear classifier relies on dot product between vectors $k(x_i,x_j)=\mathbf{x_i}^T\mathbf{x_j}$
- If every data point is mapped into high-dimensional space via some transformation Φ: $\mathbf{x} \rightarrow \phi(\mathbf{x})$, the dot product becomes:

$$k(\mathbf{x_i},\mathbf{x_j}) = \phi(\mathbf{x_i})^T\phi(\mathbf{x_j})$$

- A *kernel function* is some function that corresponds to an inner product in some expanded feature space.
- Example:

  2-dimensional vectors x=[x_1 x_2]; let $k(\mathbf{x_i},\mathbf{x_j})=(1 + \mathbf{x_i}^T\mathbf{x_j})^2$,

  Need to show that $K(\mathbf{x_i},\mathbf{x_j}) = \phi(\mathbf{x_i})^T\phi(\mathbf{x_j})$:

$$k(\mathbf{x_i},\mathbf{x_j})=(1 + \mathbf{x_i}^T\mathbf{x_j})^2,$$

$$= 1+ x_{i1}^2x_{j1}^2 + 2\, x_{i1}x_{j1}\, x_{i2}x_{j2}+ x_{i2}^2x_{j2}^2 + 2x_{i1}x_{j1} + 2x_{i2}x_{j2}$$

$$= [1\ \ x_{i1}^2\ \sqrt{2}\, x_{i1}x_{i2}\ \ x_{i2}^2\ \sqrt{2}x_{i1}\ \sqrt{2}x_{i2}]^T [1\ \ x_{j1}^2\ \sqrt{2}\, x_{j1}x_{j2}\ \ x_{j2}^2\ \sqrt{2}x_{j1}\ \sqrt{2}x_{j2}]$$

$$= \phi(\mathbf{x_i})^T\phi(\mathbf{x_j}), \quad \text{where } \phi(\mathbf{x}) = [1\ \ x_1^2\ \sqrt{2}\, x_1x_2\ \ x_2^2\ \ \sqrt{2}x_1\ \sqrt{2}x_2]$$

What Functions are Kernels?

- For many functions $k(\mathbf{x}_i,\mathbf{x}_j)$ checking that

  $$k(\mathbf{x}_i,\mathbf{x}_j)= \phi(\mathbf{x}_i)^T\phi(\mathbf{x}_j)$$ can be cumbersome.
- Mercer's theorem: Every semi-positive definite symmetric function is a kernel
  - Semi-positive definite symmetric functions correspond to a semi-positive definite symmetric Gram matrix:

K =

$k(\mathbf{x}_1,\mathbf{x}_1)$	$k(\mathbf{x}_1,\mathbf{x}_2)$	$k(\mathbf{x}_1,\mathbf{x}_3)$	…	$k(\mathbf{x}_1,\mathbf{x}_N)$
$k(\mathbf{x}_2,\mathbf{x}_1)$	$k(\mathbf{x}_2,\mathbf{x}_2)$	$k(\mathbf{x}_2,\mathbf{x}_3)$		$k(\mathbf{x}_2,\mathbf{x}_N)$
…	…	…	…	…
$k(\mathbf{x}_N,\mathbf{x}_1)$	$k(\mathbf{x}_N,\mathbf{x}_2)$	$k(\mathbf{x}_N,\mathbf{x}_3)$	…	$k(\mathbf{x}_N,\mathbf{x}_N)$

Examples of Kernel Functions

- Linear: $k(\mathbf{x}_i,\mathbf{x}_j) = \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j$
- Polynomial of power p: $k(\mathbf{x}_i,\mathbf{x}_j) = (1 + \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)^p$
- Gaussian (radial-basis function network):

$$k(\mathbf{x}_i, \mathbf{x}_j) = \exp \frac{-||\mathbf{x}_i - \mathbf{x}_j||^2}{\sigma^2}$$

- Sigmoid: $k(\mathbf{x}_i,\mathbf{x}_j) = \tanh(\beta_0 \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j + \beta_1)$

SVM Formulation

- Goal - **1) Correctly classify all training data**

$$\mathbf{w}^T\phi(\mathbf{x}_n) + b \geq 1 \quad if \quad t_n = +1$$
$$\mathbf{w}^T\phi(\mathbf{x}_n) + b \leq 1 \quad if \quad t_n = -1$$

$$t_n(\mathbf{w}^T\phi(\mathbf{x}_n) + b) \geq 1$$

2) Define the Margin

$$\frac{1}{||\mathbf{w}||} min_n \left[t_n(\mathbf{w}^T\phi(\mathbf{x}_n) + b)\right]$$

3) Maximize the Margin

$$argmax_{\mathbf{w},b}\left\{\frac{1}{||\mathbf{w}||} min_n \left[t_n(\mathbf{w}^T\phi(\mathbf{x}_n) + b)\right]\right\}$$

- Equivalently written as

$$argmin_{\mathbf{w},b} \frac{1}{2}||\mathbf{w}||^2 \text{ such that } t_n(\mathbf{w}^T\phi(\mathbf{x}_n) + b) \geq 1$$

Solving the Optimization Problem

- Need to optimize a *quadratic* function subject to *linear* constraints.
- Quadratic optimization problems are a well-known class of mathematical programming problems, and many (rather intricate) algorithms exist for solving them.
- The solution involves constructing a *dual problem* where a *Lagrange multiplier* a_n is associated with every constraint in the primary problem:
- The dual problem in this case is maximized

Find $\{a_1, .., a_N\}$ such that

$$\tilde{L}(\mathbf{a}) = \sum_{n=1}^{N} a_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} t_n t_m a_n a_m k(\mathbf{x}_n, \mathbf{x}_m) \text{ maximized}$$

and $\sum_n a_n t_n = 0, \quad a_n \geq 0$

Solving the Optimization Problem

- The solution has the form:

$$\mathbf{w} = \sum_{n=1}^{N} a_n \phi(\mathbf{x}_n)$$

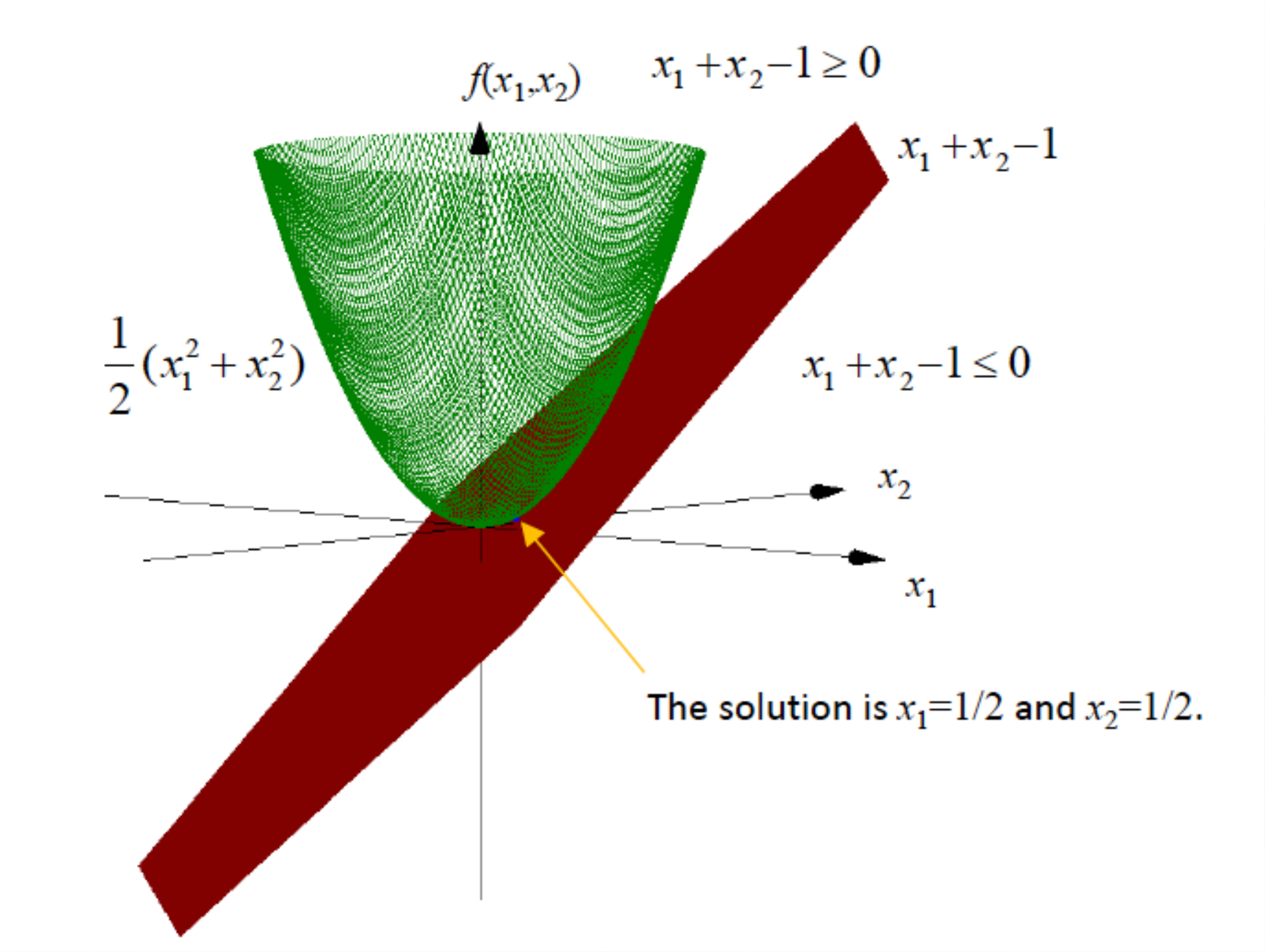
- Each non-zero a_n indicates that corresponding $\mathbf{x_n}$ is a support vector. Let S denote the set of support vectors.

$$b = y(\mathbf{x}_n) - \sum_{m \in S} a_m k(\mathbf{x}_m, \mathbf{x}_n)$$

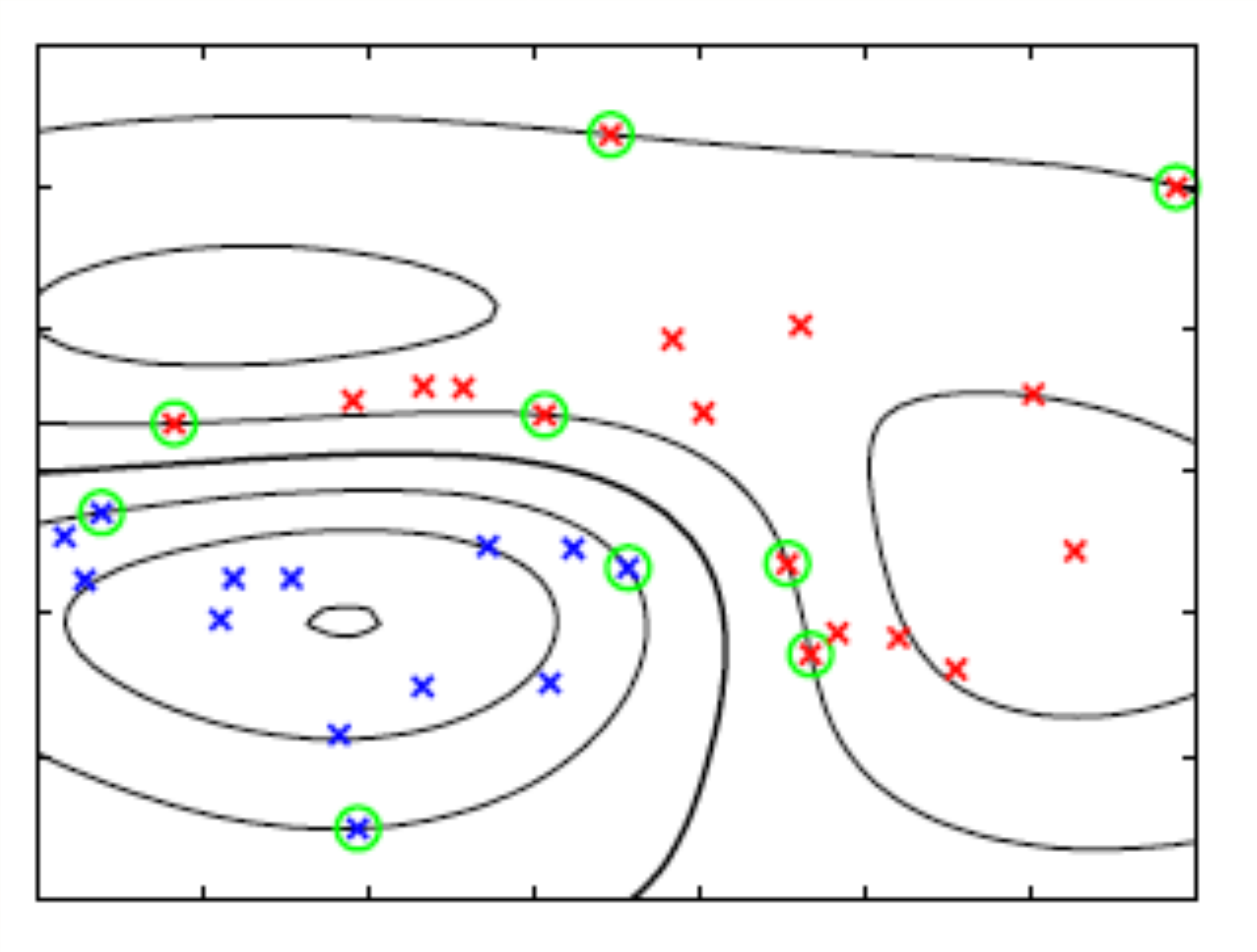
- And the classifying function will have the form:

$$y(\mathbf{x}) = \sum_{n \in S} a_n k(\mathbf{x}_n, \mathbf{x}) + b$$

Solving the Optimization Problem



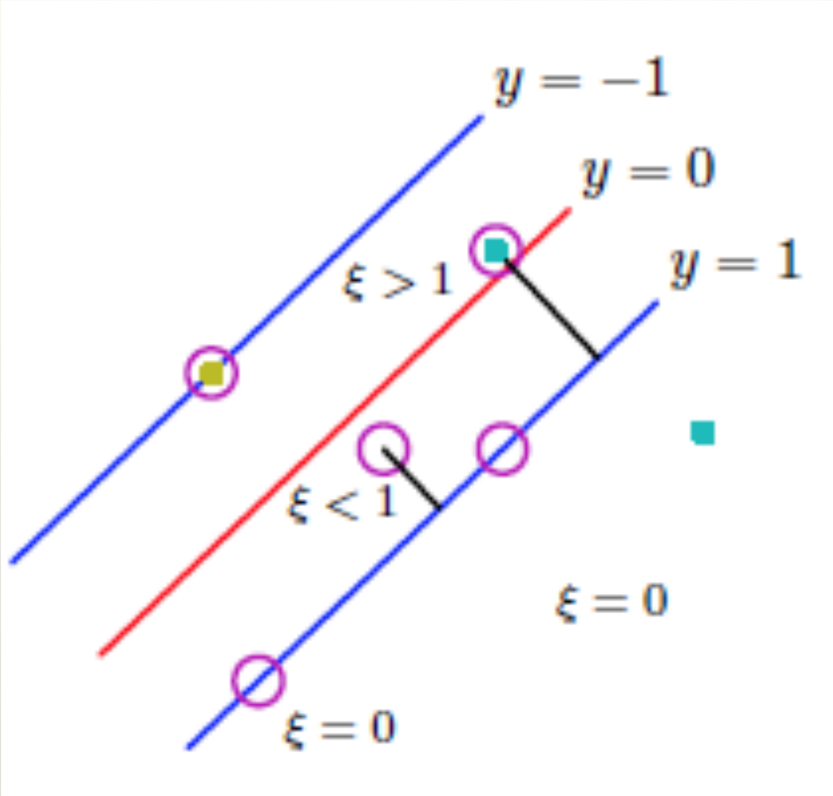
Visualizing Gaussian Kernel SVM



Overlapping class boundaries

- The classes are not linearly separable - Introducing slack variables ζ_n
- Slack variables are non-negative $\zeta_n \geq 0$
- They are defined using

$$t_n y(\mathbf{x}_n) \geq 1 - \zeta_n$$



- The upper bound on mis-classification

$$\sum_n \zeta_n$$

- The cost function to be optimized in this case

$$C \sum_n \zeta_n + \frac{1}{2}\mathbf{w}^T\mathbf{w}$$

SVM Formulation - overlapping classes

- Formulation very similar to previous case except for additional constraints

$$0 \leq a_n \leq C$$

- Solved using the dual formulation - sequential minimal optimization algorithm
- Final classifier is based on the sign of

$$y(\mathbf{x}) = \sum_{n \in S} a_n k(\mathbf{x}_n, \mathbf{x}) + b$$

Overlapping class boundaries