Wavelet Theory

$\psi(t)$: Wavelet

i) $\int_{-\infty}^{\infty} \psi(t)\, dt = 0$ 'oscillatory'

ii) $\int_{-\infty}^{\infty} |\psi(t)|^2\, dt < \infty$ finite energy.

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{t-b}{a}\right)$$

Cont. Wavelet Transform & its Inverse

$f(t) \longleftrightarrow W(a,b)$ a = scale , b = shift

$$W(a,b) = \int_{-\infty}^{\infty} f(t)\, \frac{1}{\sqrt{a}}\, \psi^*\left(\frac{t-b}{a}\right) dt \qquad \begin{matrix} 0 < a < \infty \\ -\infty < b < \infty \end{matrix}$$

$$f(t) = \frac{1}{C} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(a,b)\, \psi_{a,b}(t)\, \frac{da\, db}{|a|^2}$$

$$C \equiv \int_{-\infty}^{\infty} \frac{|\Psi(\Omega)|^2}{|\Omega|}\, d\Omega . \qquad 0 < C < \infty .$$

Dyadic Wavelet Decomposition

Let us sample 'a' at $a = 2^{-j}$ $-\infty < j < \infty$
and 'b' at $b = 2^{-j} k$

Then $\psi_{j,k}(t) = 2^{j/2}\, \psi(2^j t - k)$

Now $\quad W(a,b)$ at $\quad a = 2^{-j}$

$\qquad\qquad\qquad\qquad\quad b = k2^{-j}$

$W(2^{-j}, k2^{-j})$ will be denoted as

"d_{JK}"

ie d_{JK} = Coef of $2^{J/2}\psi(2^{J}t-k)$

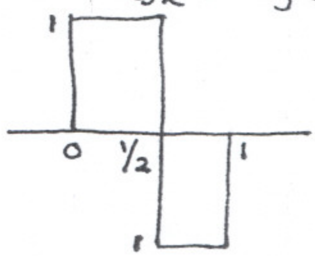
Dyadic Wavelet expansion (decomposition) of a function $f(t)$ is now given by

$$f(t) = \sum_{J}\sum_{k} d_{JK}\,\psi_{JK}(t)$$

If ψ_{JK} are orthogonal $d_{JK} = \langle f(t), \psi^{*}_{JK}\rangle$

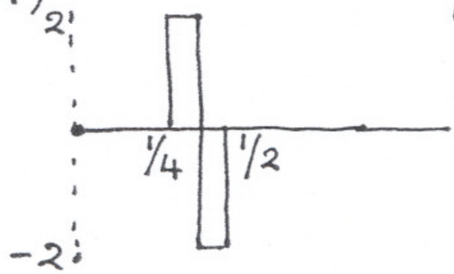
<u>Haar Wavelet</u>

$$\psi(t) = \psi_{1,0}(t)$$



$$\psi_{2,1}(t) = 2^{1}\psi(2^{2}t-1)$$

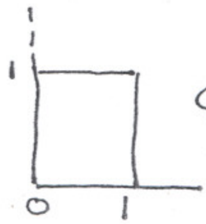
Energy $= 2^{2}\cdot\frac{1}{4} = 1$



<u>~~Haar~~ Scaling function</u>

$\phi(t)$

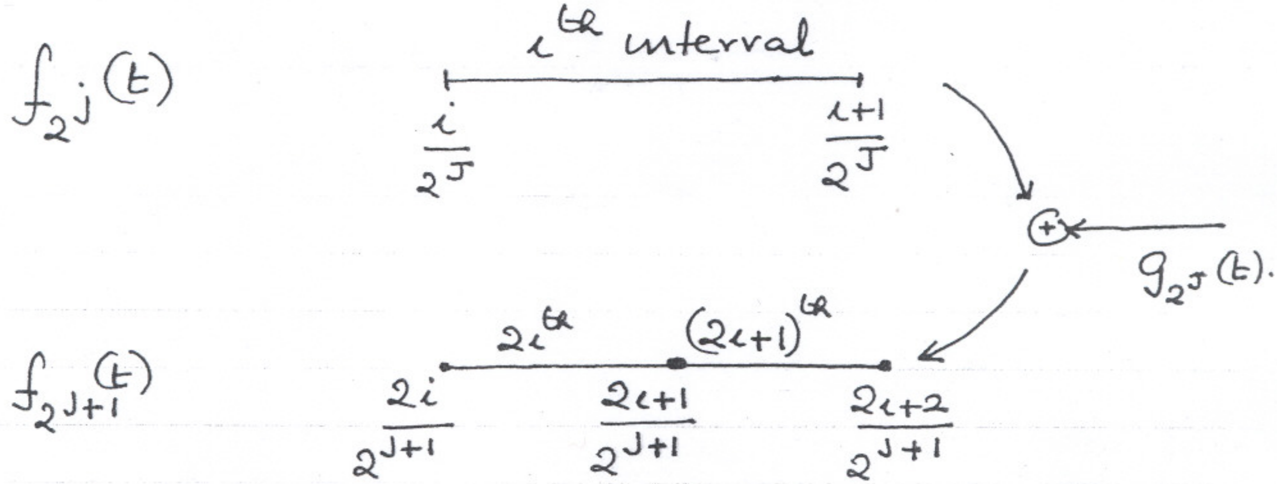
Haar Scaling function



$\phi(t)$ for Haar

Linear Piecewise Constant approx of a signal $f(t)$

This kind of approx ~~leads~~ Corresponds to Haar Wavelet / Scaling function application to $f(t)$

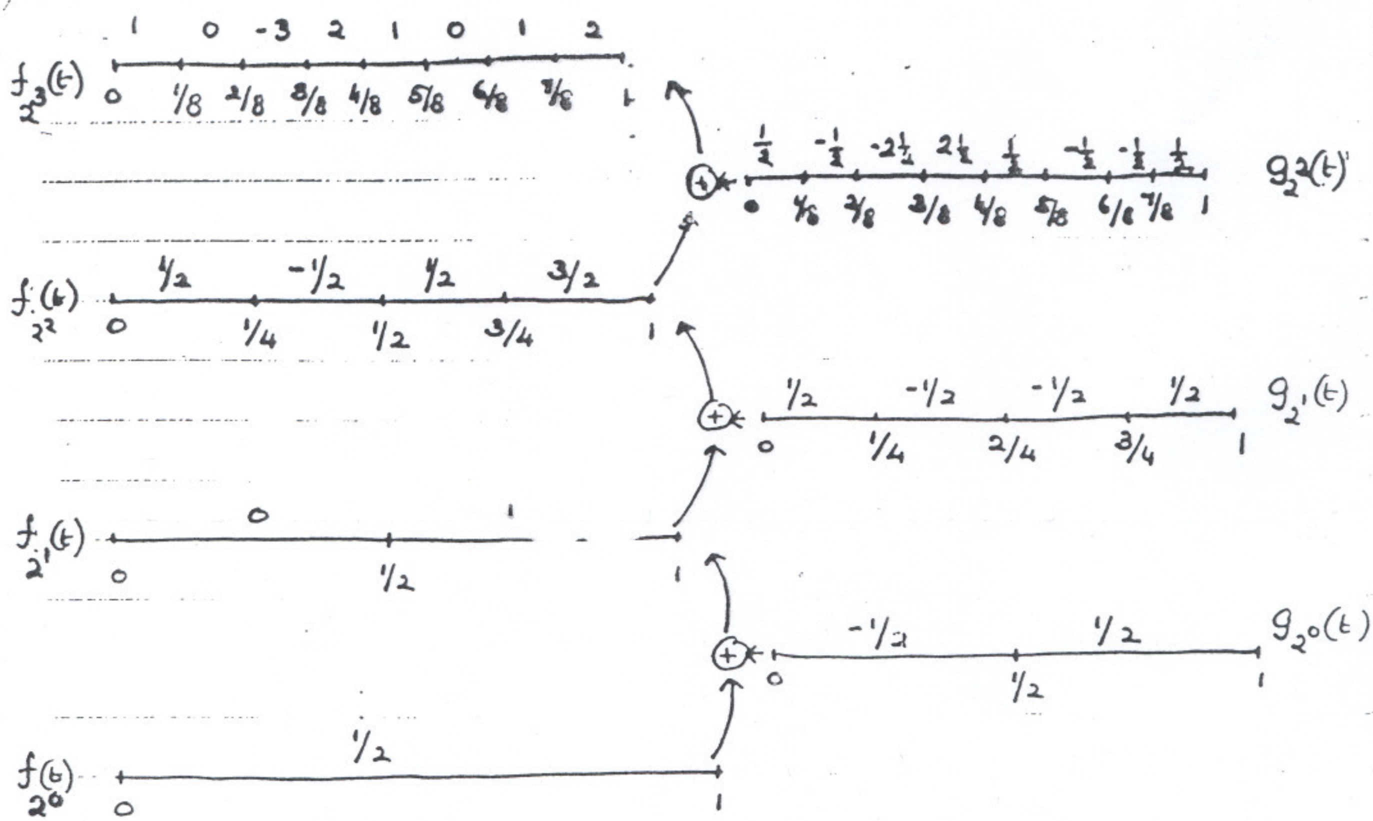


$g_{2^J}(t)$ = detail signal at level J

$f_{2^J}(t)$ = approx at level J of the signal $f(t)$

$$g_{2^J}(t) = f_{2^{J+1}}(t) - f_{2^J}(t)$$

$$\lim_{J \to \infty} f_{2^J}(t) = f(t) = \sum_{J=-\infty}^{\infty} g_{2^J}(t)$$

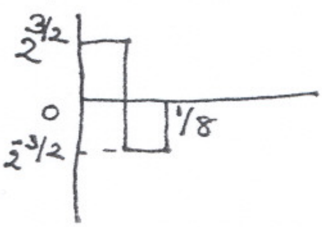


$f_{2^J}(t)$ Can be expressed as linear Combination of ψ_{Jk}

$g_{2^J}(t)$ " " " ϕ_{Jk}

Eg: $g_{2^3}(t)$ Can be expressed as a l.c of

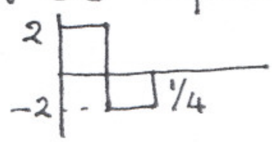
$\psi_{30}(t)$



and its translat ie $\psi_{31}, \psi_{32}, \psi_{33}$

$g_{2^2}(t) \longrightarrow$ Can be expressed as l.c of

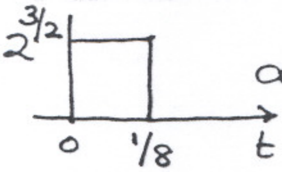
$\psi_{20}(t)$



and its translates ie $\psi_{21}, \psi_{22}, \psi_{23}$

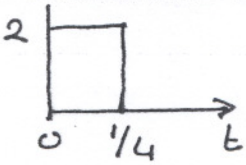
Similarly

$f_2{}^3(t)$ Can be expressed as a l.c of

$\phi_{30}(t)$  and its translates $\phi_{31}, \phi_{32}, \dots$

$\phi_{30}(t) = 2^{3/2}\,\phi(2^3 t)$

$f_2{}^2(t)$ Can be expressed as a l.c of

$\phi_{20}(t)$  and its translates $\phi_{21}, \phi_{22}, \phi_{23}, \dots$

<u>**Decomposition**</u>

$$\begin{tikzcd}[column sep=small]
f_2{}^3(t) \arrow[r] \arrow[dr, no head] & g_2{}^2(t) & & \\
& f_2{}^2(t) \arrow[r, no head] \arrow[dr, no head] & g_2{}^1(t) & \\
& & f_2{}^1(t) \arrow[r, no head] \arrow[dr, no head] & g_2{}^0(t) \quad \text{detail} \\
& & & f_2{}^0(t) \quad \text{approx}
\end{tikzcd}$$

$$f_2{}^3(t) = g_2{}^2(t) + g_2{}^1(t) + g_2{}^0(t) + f_2{}^0(t)$$

$$f_2^3(t) = [d_{20}\psi_{20} + d_{21}\psi_{21} + d_{22}\psi_{22} + d_{23}\psi_{23}]$$
$$+ [d_{10}\psi_{10} + d_{11}\psi_{11}] + [d_{00}\psi_{00}] + c_{00}\phi_{00}$$

$$[d_{20}, d_{21}, d_{22}, d_{23}] = \left[\frac{1}{4}, -\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}\right]$$

$$[d_{10}, d_{11}] = \left[+\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right]$$

$$d_{00} = -\frac{1}{2} \qquad c_{00} = \frac{1}{2}$$

$f_2^3 \to \left[\frac{1}{2\sqrt{2}}, 0, -\frac{3}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ $\{\phi_{3k}\}$

- $\left[\frac{1}{4}, -\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}\right]$ $\{\psi_{2k}\}$
- $\left[\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right]$ $\{\phi_{2k}\}$
  - $\left[1/2\sqrt{2}, -1/2\sqrt{2}\right]$ $\{\psi_{1k}\}$
  - $\left[0, \frac{1}{\sqrt{2}}\right]$ $\{\phi_{1k}\}$
    - $\left[-\frac{1}{2}\right]$ $\{\psi_{0k}\}$
    - $\left[\frac{1}{2}\right]$ $\{\phi_{0k}\}$

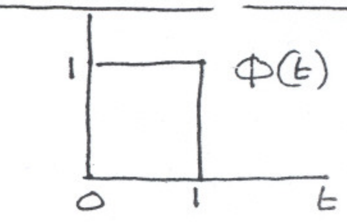
$$\phi_{Jk} = 2^{J/2}\phi(2^J t - k)$$
$$\psi_{Jk} = 2^{J/2}\psi(2^J t - k)$$

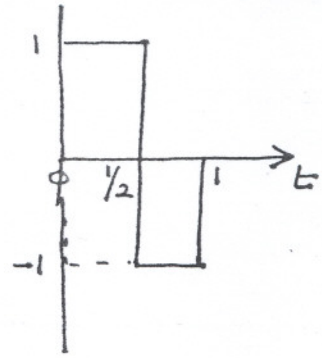
$[c_{30}, c_{31}, c_{32}, c_{33}, c_{34}, c_{35}, c_{36}, c_{37}, c_3]$

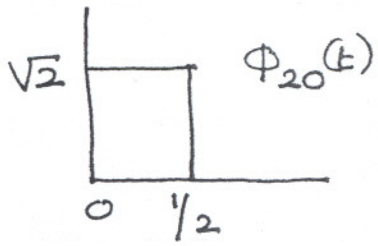
- $\boxed{G}$ — $[d_{20}, d_{21}, d_{22}, d_{23}]$
- $\boxed{H}$
  - $\boxed{G}$ — $[d_{10}, d_{11}]$
  - $\boxed{H}$
    - $[d_{00}]$
    - $[c_{00}]$

$$\phi(t) = \sum h(k)\sqrt{2}\,\phi(2t-k)$$

Relation between Haar Wavelet & Scaling function







$$\phi(t) = \phi_{00}(t) = \frac{1}{\sqrt{2}}\,\phi_{20}(t) + \frac{1}{\sqrt{2}}\,\phi_{21}(t)$$

(A) $\phi(t) = \sum_k h(k)\,\phi_{2k}(t)$ (in general) $\qquad h(0) = h(1) = 1/\sqrt{2}$

$$\psi(t) = \frac{1}{\sqrt{2}}\,\phi_{20}(t) - \frac{1}{\sqrt{2}}\,\phi_{21}(t)$$

(B) $\psi(t) = \sum_k g(k)\,\phi_{2k}(t)$ (in general) $\qquad$ here $k = 0, 1$.

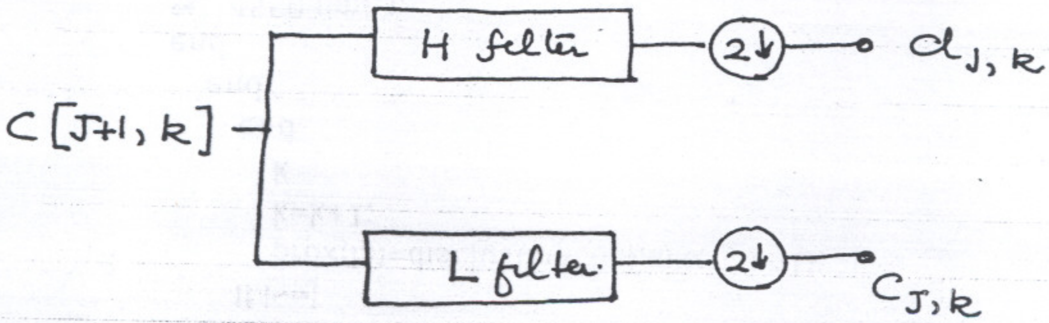
One can show $\quad g(k) = (-1)^k\, h(-k+1)$

eg: $g(0) = h(1) = 1/\sqrt{2}$

$g(1) = -1 \times h(0) = -1/\sqrt{2}$.

A & B are called Dilation or two scale equation

$$C(J+1,k) \begin{cases} d_{Jk} & k = \cdots 0,1,2 \\ c_{Jk} & k = -1,0,1,\ldots \end{cases}$$



H filter : $h_\psi(-n)$

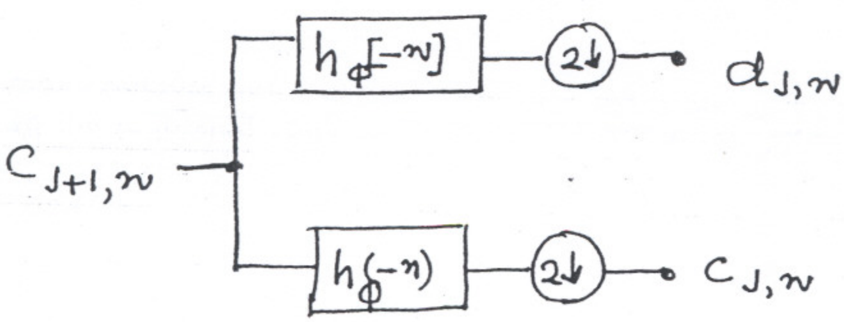
L filter $h_\phi(-n)$

$h_\phi[n] = \{h_0, h_1\}$

$h_\psi[n] = \{g_0, g_1\}$

$$d(J,k) = \sum_m h_\psi(m-2k)\, c(J+1,k)$$

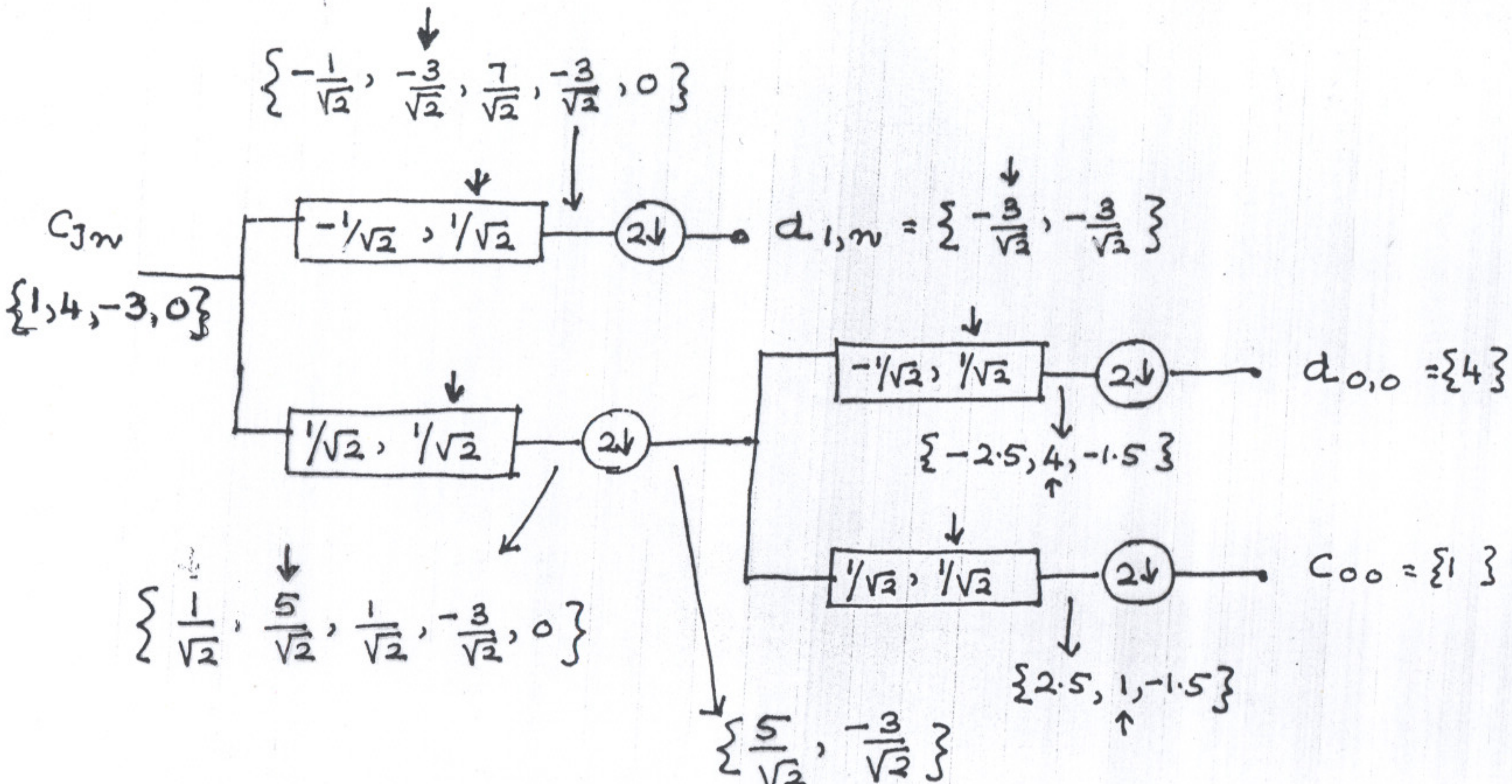
$$c(J,k) = \sum_m h_\phi(m-2k)\, c(J+1,k)$$

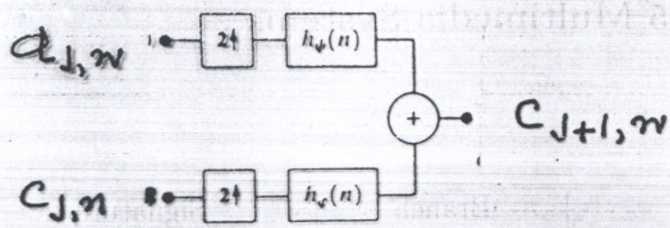


$n = -1,0,1,2,\ldots$

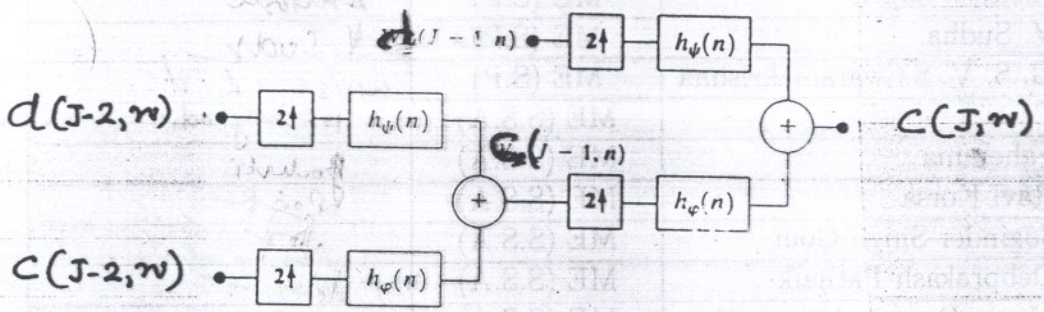
$n = -1,0,1,2,\ldots$

W-9

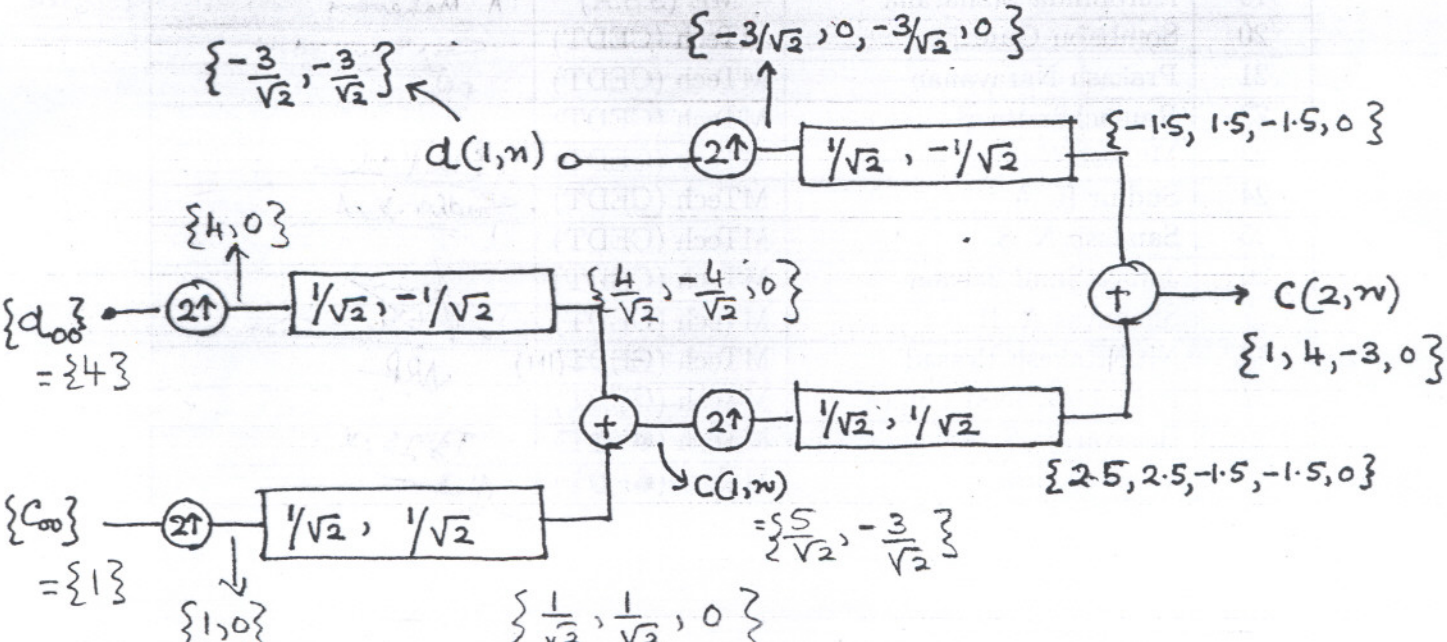

$C_{J,n}$
$\{1, 4, -3, 0\}$
$-1/\sqrt{2}, 1/\sqrt{2}$
$1/\sqrt{2}, 1/\sqrt{2}$
$\{-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, \frac{7}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 0\}$
$2\downarrow$
$d_{1,n} = \{-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\}$
$\{\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}, 0\}$
$\{\frac{5}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\}$
$\{-2.5, 4, -1.5\}$
$\{2.5, 1, -1.5\}$
$d_{0,0} = \{4\}$
$C_{00} = \{1\}$



Synthesis filter.



2-stage synthesis filter.



2-scale Inverse Wavelet Transform.

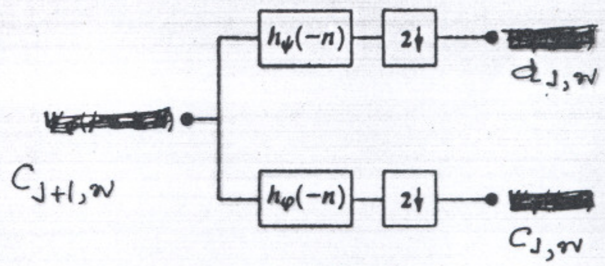
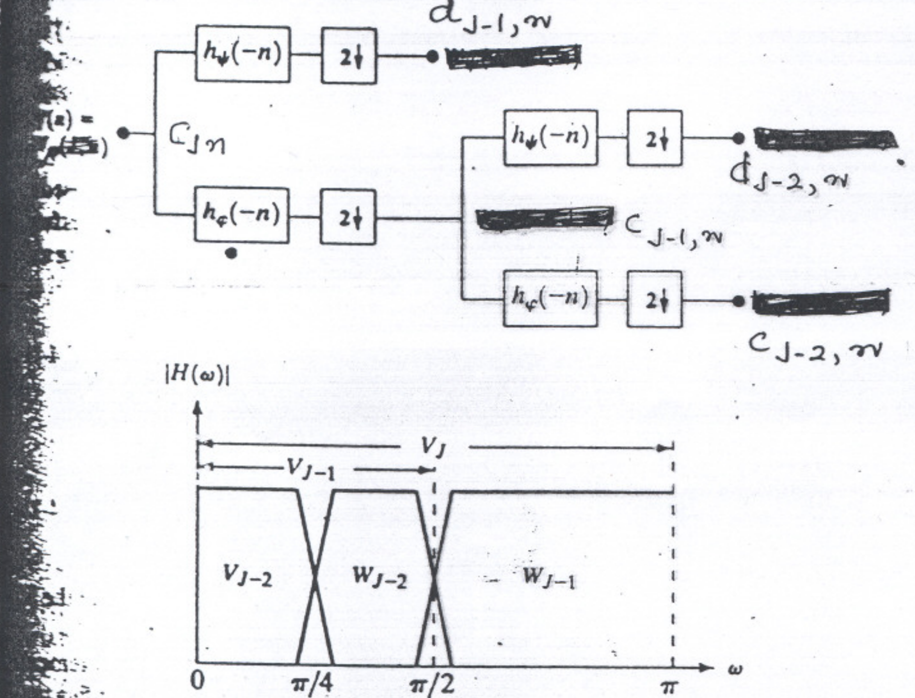


FIGURE 7.15 An FWT analysis bank.



a
b

FIGURE 7.16 (a) A two-stage or two-scale FWT analysis bank and (b) its frequency splitting characteristics.

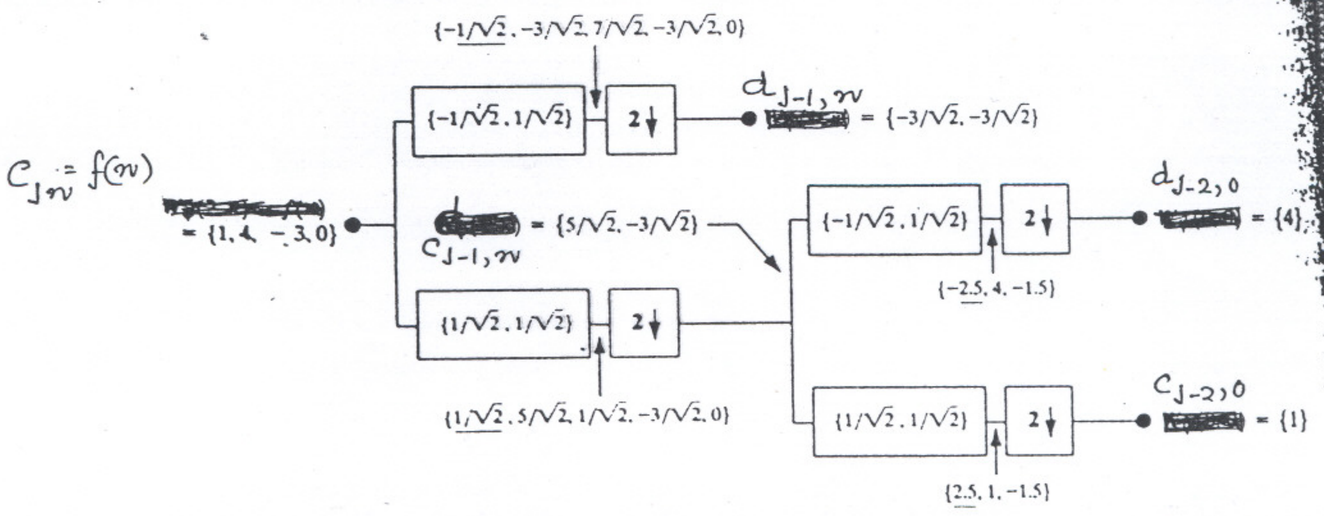


FIGURE 7.17 Computing a two-scale fast wavelet transform of sequence $\{1, 4, -3, 0\}$ using Haar scaling and wavelet vectors.

$-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

$0 \quad -3 \quad 4 \quad 1$

$\frac{4}{\sqrt{2}}, \frac{3}{\ }$

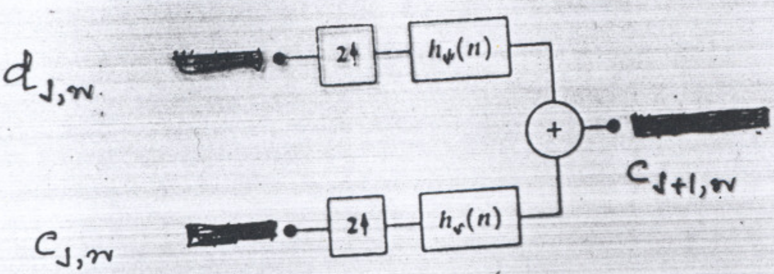
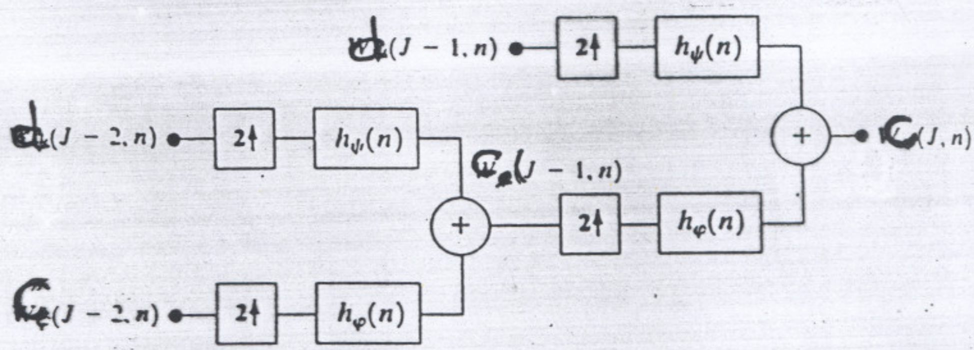
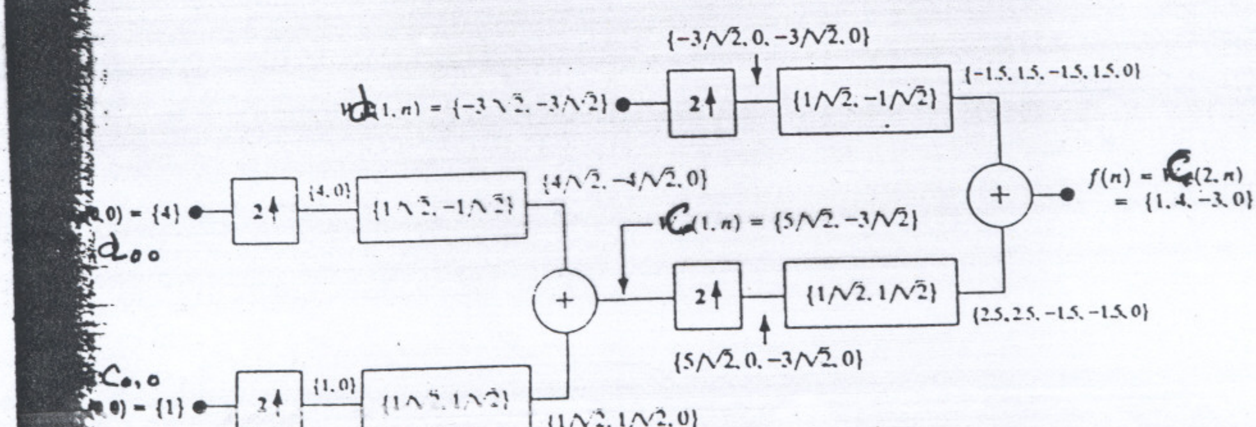


FIGURE 7.18 The FWT^{-1} synthesis filter bank.

FIGURE 7.19 A two-stage or two-scale FWT^{-1} synthesis bank.





URE 7.20 Computing a two-scale inverse fast wavelet transform of sequence $\{1, 4, -1.5\sqrt{2}, -1.5\sqrt{2}\}$ with ar scaling and wavelet vectors.