Wavelet Theory

$\psi(t)$: Wavelet

i) $\int_{-\infty}^{\infty} \psi(t)\, dt = 0$ 'oscillatory'

ii) $\int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty$ finite energy.

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{t-b}{a}\right)$$

Cont. Wavelet Transform & its Inverse

$f(t) \longleftrightarrow W(a,b)$ a = scale , b = shift

$$W(a,b) = \int_{-\infty}^{\infty} f(t)\, \frac{1}{\sqrt{a}}\, \psi^*\left(\frac{t-b}{a}\right) dt \qquad 0 < a < \infty, \; -\infty < b < \infty$$

$$f(t) = \frac{1}{C} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(a,b)\, \psi_{a,b}(t)\, \frac{da\, db}{|a|^2}$$

$$C \equiv \int_{-\infty}^{\infty} \frac{|\Psi(\Omega)|^2}{|\Omega|}\, d\Omega. \qquad 0 < C < \infty.$$

Dyadic Wavelet Decomposition

Let us sample 'a' at $a = 2^{-j}$ $-\infty < j < \infty$
and 'b' at $b = 2^{-j} k$

Then $\psi_{j,k}(t) = 2^{j/2}\, \psi(2^j t - k)$

Now $W(a,b)$ at $a = 2^{-j}$, $b = k2^{-j}$

$W(2^{-j}, k2^{-j})$ will be denoted as

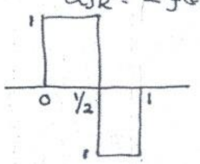
"d_{JK}"

ie d_{JK} = Coef of $2^{J/2}\,\psi(2^J t - k)$

Dyadic Wavelet expansion (decomposition) of a function $f(t)$ is now given by

$$f(t) = \sum_J \sum_k d_{JK}\, \psi_{JK}(t)$$

If ψ_{JK} are orthogonal $d_{JK} = \langle f(t), \psi^*_{JK} \rangle$

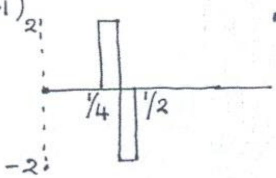


Haar Wavelet

$$\psi(t) = \psi_{0,0}(t)$$

$$\psi_{2,1}(t) = 2^1\,\psi(2^2 t - 1)$$

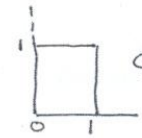
Energy $= 2^2 \cdot \frac{1}{4} = 1$



~~Haar~~ Scaling function

$\phi(t)$

Haar Scaling function

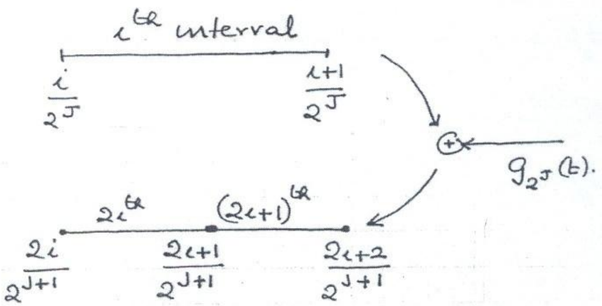


$\phi(t)$ for Haar

Linear Piecewise Constant approx of a signal $f(t)$

This kind of approx ~~leads~~ Corresponds to Haar Wavelet/ Scaling function application to $f(t)$

$f_{2^J}(t)$



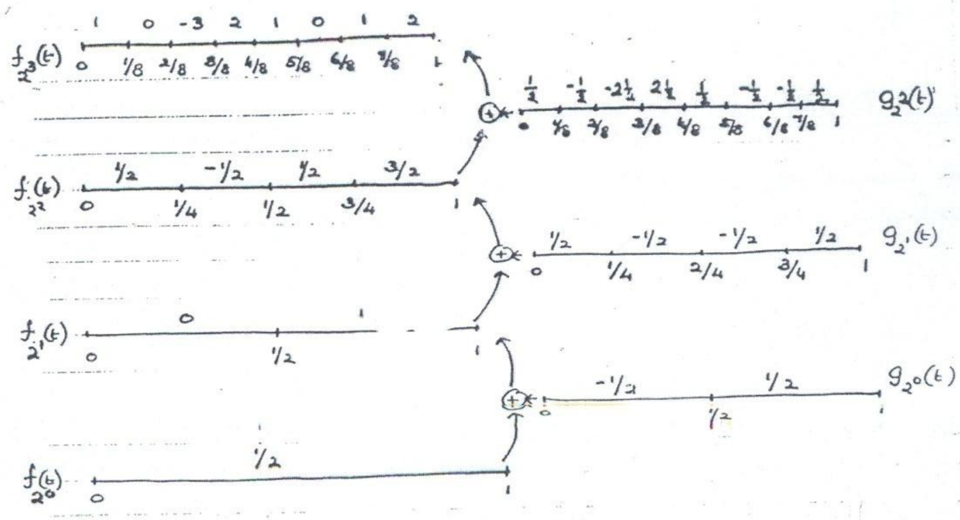
$f_{2^{J+1}}(t)$

$g_{2^J}(t)$ = detail signal at level J

$f_{2^J}(t)$ = approx at level J of the signal $f(t)$

$$g_{2^J}(t) = f_{2^{J+1}}(t) - f_{2^J}(t)$$

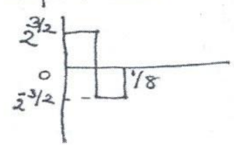
$$\lim_{J \to \infty} f_{2^J}(t) \equiv f(t) \equiv \sum_{J=-\infty}^{\infty} g_{2^J}(t)$$



$f_{2^J}(t)$ Can be expressed as linear Combination of φ_{Jk}

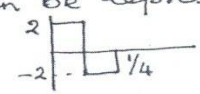
$g_{2^J}(t)$ " " " Φ_{Jk}

Eg: $g_{2^3}(t)$ Can be expressed as a l.c of

$\psi_{30}(t)$ 

and its translat ie $\psi_{31}, \psi_{32}, \psi_{33}$

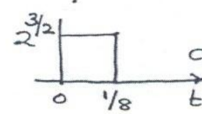
$g_{2^2}(t) \longrightarrow$ Can be expressed as l.c of

$\psi_{20}(t)$ 

and its translates ie $\psi_{21}, \psi_{22}, \psi_{23}$

Similarly

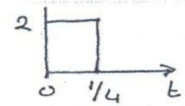
$f_2^3(t)$ Can be expressed as a l.c of

$\phi_{30}(t)$ 

and its translates $\phi_{31}, \phi_{32}, \dots$

$\phi_{30}(t) = 2^{3/2}\phi(2^3 t)$

$f_2^2(t)$ Can be expressed as a l.c of

$\phi_{20}(t)$ 

and its translates $\phi_{21}, \phi_{22}, \phi_{23}, \dots$

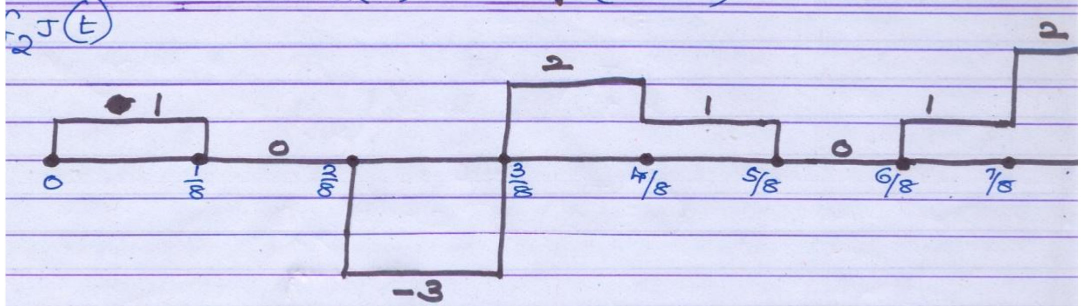
Decomposition

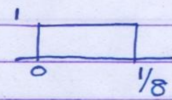
$$f_2^3(t) \to \begin{cases} g_2^2(t) \\ f_2^2(t) \to \begin{cases} g_2^1(t) \\ f_2^1(t) \to \begin{cases} g_2^0(t) & \text{detail} \\ f_2^0(t) & \text{approx} \end{cases} \end{cases} \end{cases}$$

$$f_2^3(t) = g_2^2(t) + g_2^1(t) + g_2^0(t) + f_2^0(t)$$

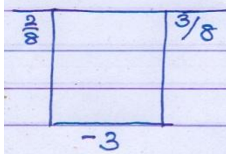
$$\phi_{JK}(t) = 2^{J/2}\phi(2^J t - k)$$
$$\psi_{JK}(t) = 2^{J/2}\psi(2^J t - k)$$

N·5 1

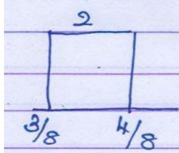




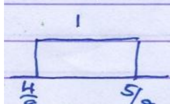
$$= \phi(2^3 t - 0) = 2^{-3/2}\cdot 2^{3/2}\phi(2^3 t - 0) = 2^{-3/2}\phi_{30}(t)$$
$$= \frac{1}{2\sqrt{2}}\phi_{30}(t)$$



$$= -3\phi(2^3 t - 2) = -3\cdot 2^{-3/2}\cdot 2^{+3/2}\phi(2^3 t - 2)$$
$$= -\frac{3}{2\sqrt{2}}\phi_{32}(t)$$



$$= 2\phi(2^3 t - 3) = 2\cdot 2^{-3/2}\, 2^{+3/2}\phi(2^3 t - 2)$$
$$= \frac{1}{\cancel{2}\sqrt{2}}\phi_{32}(t)$$



$$= 2^{-3/2}\, 2^{3/2}\phi(2^3 t - 3) = \frac{1}{2\sqrt{2}}\phi_{33}(t)$$

$$\frac{1}{2\sqrt{2}},\ 0,\ -\frac{3}{2\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \frac{1}{2\sqrt{2}},\ 0,\ \frac{1}{2\sqrt{2}},\ \frac{1}{\sqrt{2}}$$

$$f_{2J}(t) = \frac{1}{2\sqrt{2}}\phi_{30}(t) + 0\cdot\phi_{31}(t) + \left(\frac{-3}{2\sqrt{2}}\right)\phi_{32}(t)$$

$$+\frac{1}{2\sqrt{2}}\phi_{33}(t) + \frac{1}{2\sqrt{2}}\phi_{34}(t) + 0\cdot\phi_{35}(t)$$

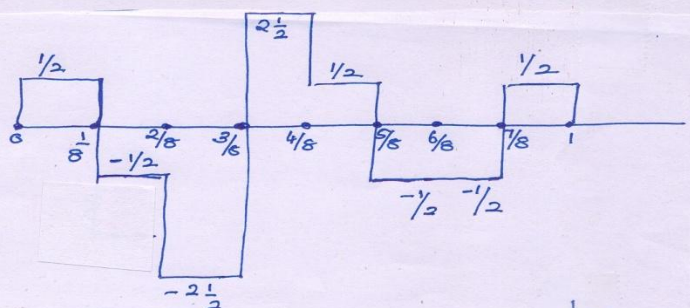
$$+\frac{1}{2\sqrt{2}}\phi_{36}(t) + \frac{1}{\sqrt{2}}\phi_{37}(t)$$

$$= \left\{\frac{1}{2\sqrt{2}}, 0, \frac{3}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

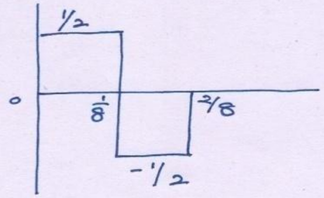
$$f_{2J}(t) = \sum_k c_{3k}\,\phi_{3k}$$

$$f_{2J}(t) = \sum_k c_{Jk}\,\phi_{Jk}$$

$g_{2}2(t) =$

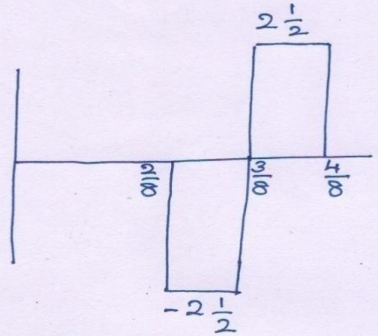


L



$$\frac{1}{2}\,\psi(4t-1) = \frac{1}{4}\cdot 2^{1}\,\psi(2^{2}t-0)$$

$$= \frac{1}{4}\,\psi_{20}(t)$$



$$= -\frac{5}{4}\cdot 2^{1}\,\psi(2^{2}t-1)$$

$$= -\frac{5}{4}\,\psi_{21}(t)$$

Note

$f_{2^J}(t) = $ l.c of $\phi_{JK}(t)$ $\quad k = \{-\infty, \cdots 0, 1, 2 \cdots\}$

$g_{2^J}(t) = $ l.c of $\psi_{JK}(t)$ $\quad$ " $\quad$ "

Example

$$f_{2^3}(t) = \sum_k c_{3k}\,\phi_{3k} \;\Big\langle\; \begin{array}{l} g_{2^2}(t) = \sum_k d_{2k}\,\psi_{2k}(t) \\ f_{2^2}(t) = \sum_k c_{2k}\,\phi_{2k}(t) \;\Big\langle\; \begin{array}{l} g_{2^1}(t) \\ f_{2^1}(t) \end{array} \end{array}$$

W-8

$$f_2^3(t) = [d_{20}\psi_{20} + d_{21}\psi_{21} + d_{22}\psi_{22} + d_{23}\psi_{23}]$$
$$+ [d_{10}\psi_{10} + d_{11}\psi_{11}] + [d_{00}\psi_{00}] + C_{00}\phi_{00}$$

$$[d_{20}, d_{21}, d_{22}, d_{23}] = \left[\frac{1}{4}, -\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}\right]$$

$$[d_{10}, d_{11}] = \left[+\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right]$$

$$d_{00} = -\frac{1}{2} \qquad C_{00} = \frac{1}{2}$$

$$f_2^3 \rightarrow \left[\frac{1}{2\sqrt{2}}, 0, -\frac{3}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \{\phi_{3k}(t)\}$$

- $\left[\frac{1}{4}, -\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}\right]$ $\{\psi_{2k}\}$
- $\left[\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right]$ $\{\phi_{2k}\}$
  - $[1/2\sqrt{2}, -1/2\sqrt{2}]$ $\{\psi_{1k}\}$
  - $\left[0, \frac{1}{\sqrt{2}}\right]$ $\{\phi_{1k}\}$
    - $[-1/2]$ $\{\psi_{0k}\}$
    - $\left[\frac{1}{2}\right]$ $\{\phi_{0k}\}$

$$\phi_{Jk} = 2^{J/2}\phi(2^J t - k)$$
$$\psi_{Jk} = 2^{J/2}\psi(2^J t - k)$$

$[c_{30}, c_{31}, c_{32}, c_{33}, c_{34}, c_{35}, c_{36}, c_{37}, c_3]$

- $\boxed{G}$ — $[d_{20}, d_{21}, d_{22}, d_{23}]$
- $\boxed{H}$
  - $\boxed{G}$ — $[d_{10}, d_{11}]$
  - $\boxed{H}$
    - $[d_{00}]$
    - $[c_{00}]$

$$\phi(t) = \sum h(k)\sqrt{2}\,\phi(2t-k)$$

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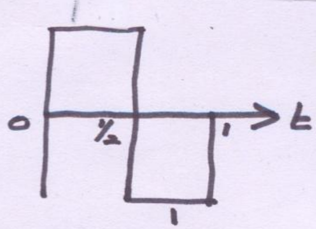
One level Decomposition

$$f_{2^J}(t) \longrightarrow \begin{cases} g_{2^{J-1}}(t) & \text{detail} \\ f_{2^{J-1}}(t) & \text{approx} \end{cases}$$

$$\{c_{Jk}\} \rightarrow \begin{cases} \{d_{J-1,k}\} \\ \{c_{J-1,k}\} \rightarrow \begin{cases} \{d_{J-2,k}\} \\ \{c_{J-2,k}\} \end{cases} \end{cases}$$



$$\psi(t) = \phi(2t) - \phi(2t-1)$$

$$= \frac{1}{\sqrt{2}} \cdot 2^{1/2}\phi(2t) - \frac{1}{\sqrt{2}} \cdot 2^{1/2}\phi(t-1)$$

$$\psi_{00}(t) = \frac{1}{\sqrt{2}}\phi_{10}(t) - \frac{1}{\sqrt{2}}\phi_{11}(t)$$

$$\phi_{00}(t) = \frac{1}{\sqrt{2}}\phi_{10}(t) + \frac{1}{\sqrt{2}}\phi_{11}(t)$$

1) $\psi(t) = \sum_k g(k)\, \phi_{1k}(t)$

2) $\phi(t) = \sum_k h(k)\, \phi_{1k}(t) = \frac{1}{\sqrt{2}} \phi_{10}(t) + \frac{1}{\sqrt{2}} \phi_{11}(t)$

$$h(0) = h(1) = \frac{1}{\sqrt{2}}$$

$$\psi(t) = \sum_k g(k)\, \phi_{1k}(t)$$

$$= \frac{1}{\sqrt{2}} \phi_{10}(t) - \frac{1}{\sqrt{2}} \phi_{11}(t)$$

$$g(0) = 1/\sqrt{2}, \quad g(1) = -\frac{1}{\sqrt{2}}$$

One Can Show

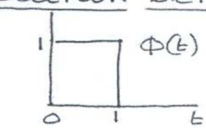
$$g(k) = (-1)^k h(1-k)$$

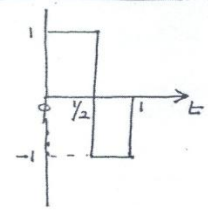
$$g(0) = h(1) = \frac{1}{\sqrt{2}}$$

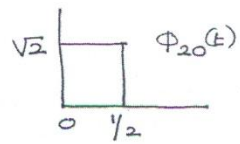
$$g(1) = -h(0) = -\frac{1}{\sqrt{2}}$$

(1) + (2) $\left.\begin{array}{l}\text{Dilation}\\ \text{Two Scale}\end{array}\right\}$ eqn.

Relation between Haar Wavelet & Scaling function







$$\phi(t) = \phi_{00}(t) = \frac{1}{\sqrt{2}}\,\phi_{20}(t) + \frac{1}{\sqrt{2}}\,\phi_{21}(t)$$

(A) $\quad \phi(t) = \sum_k h(k)\,\phi_{2k}(t)$ (in general) $\qquad h(0) = h(1) = 1/\sqrt{2}$

$$\psi(t) = \frac{1}{\sqrt{2}}\,\phi_{20}(t) - \frac{1}{\sqrt{2}}\,\phi_{21}(t)$$

(B) $\quad \psi(t) = \sum_k g(k)\,\phi_{2k}(t)$ (in general) $\qquad$ here $k = 0, 1.$

One can show $\quad g(k) = (-1)^k\, h(-k+1)$

eg: $g(0) = h(1) = 1/\sqrt{2}$

$g(1) = -1 \times h(0) = -1/\sqrt{2}.$

A & B are called Dilation or two scale equation

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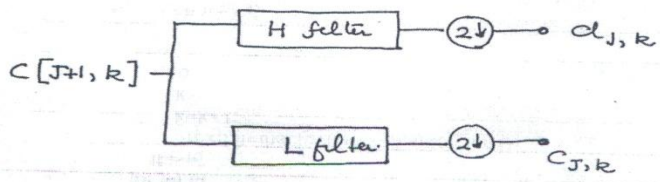
~~W-8~~

$$C(J+1, k) \begin{cases} d_{Jk} & k = \cdots 0, 1, 2 \\ c_{Jk} & k = -1, 0, 1, \ldots \end{cases}$$

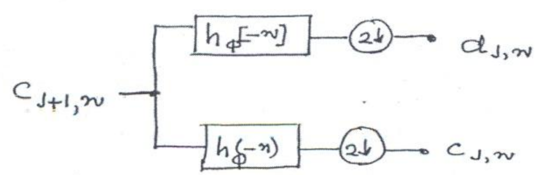


H filter : $h_\psi(-n)$ $h_\phi[n] = \{h_0, h_1\}$

L filter $h_\phi(-n)$ $h_\psi[n] = \{g_0, g_1\}$

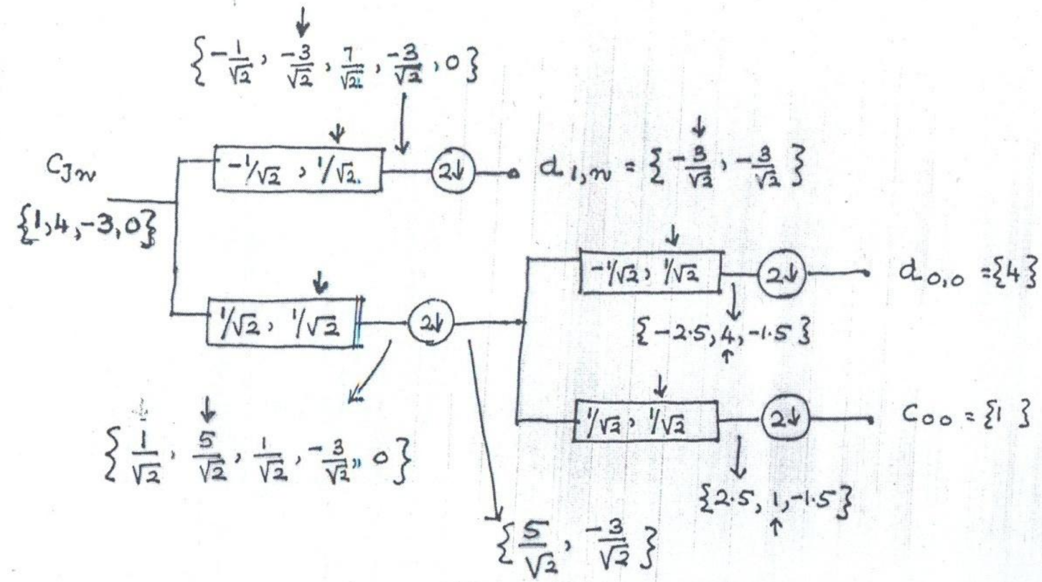
$$d(J, k) = \sum_m h_\psi(m - 2k)\, c(J+1, k)$$

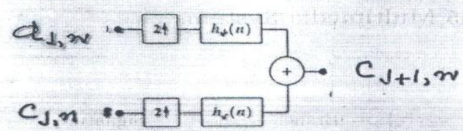
$$c(J, k) = \sum_m h_\phi(m - 2k)\, c(J+1, k)$$



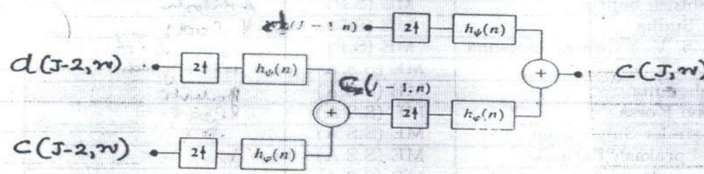
$n = -1, 0, 1, 2, \ldots$

$n = -1, 0, 1, 2, \ldots$

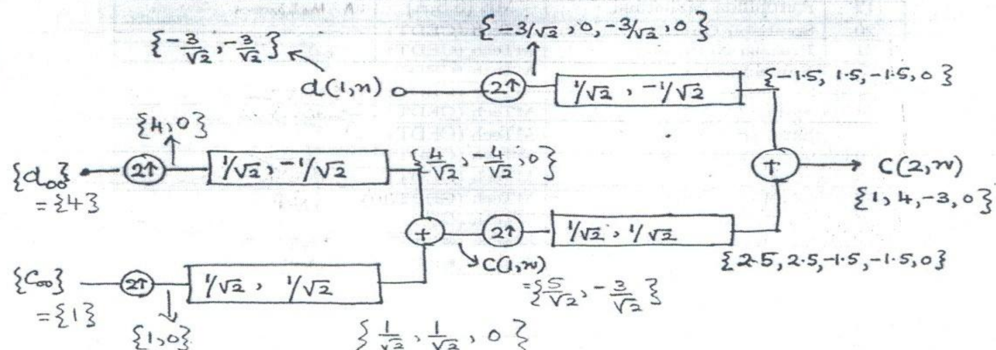

{-1/√2, -3/√2, 7/√2, -3/√2, 0}
-1/√2, 1/√2
2↓
d1,n = {-3/√2, -3/√2}
CJn
{1, 4, -3, 0}
-1/√2, 1/√2
2↓
d0,0 = {4}
1/√2, 1/√2
2↓
{-2.5, 4, -1.5}
1/√2, 1/√2
2↓
C00 = {1}
{1/√2, 5/√2, 1/√2, -3/√2, 0}
{2.5, 1, -1.5}
{5/√2, -3/√2}



Synthesis filter.



2-stage synthesis filter.



2-scale Inverse Wavelet Transform.

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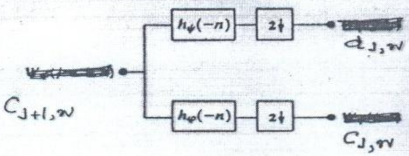
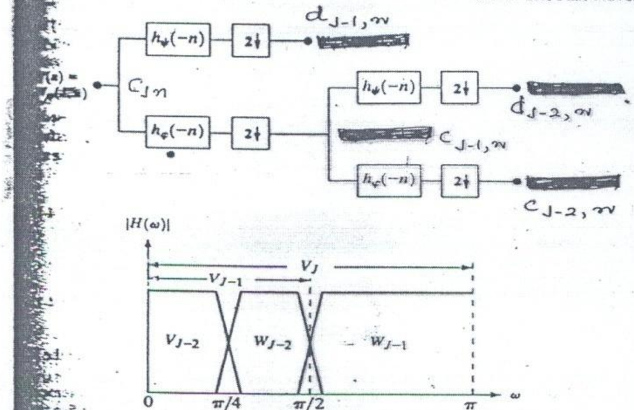
FIGURE 7.15 An FWT analysis bank.





a
b

FIGURE 7.16 (a) A two-stage or two-scale FWT analysis bank and (b) its frequency splitting characteristics.

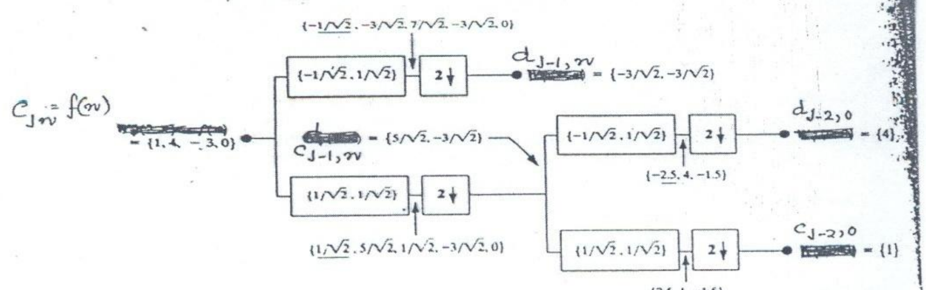


FIGURE 7.17 Computing a two-scale fast wavelet transform of sequence {1, 4, −3, 0} using Haar scaling and wavelet vectors.

$-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

0 −3 4 1

$\frac{4}{\sqrt{2}}$

16

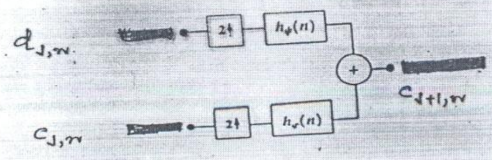
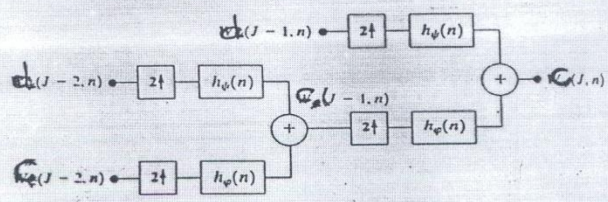
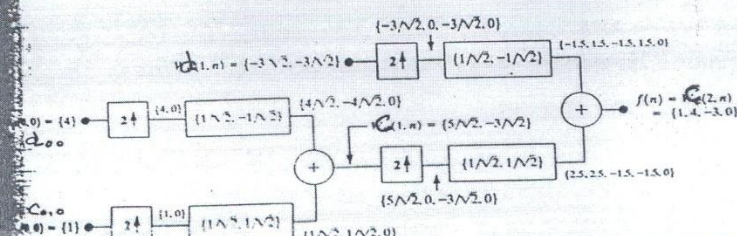


FIGURE 7.18 The FWT^{-1} synthesis filter bank.

FIGURE 7.19 A two-stage or two-scale FWT^{-1} synthesis bank.





URE 7.20 Computing a two-scale inverse fast wavelet transform of sequence $\{1, 4, -1.5\sqrt{2}, -1.5\sqrt{2}\}$ with ar scaling and wavelet vectors.