*E9 205 Machine Learning for Signal Processing*

# ML, MAP, MMSE and Gaussian Modeling

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# Recap …

- Decision Theory
  - Inference problem
    - Finding the joint density $p(\mathbf{x}, \mathbf{t})$
  - Decision problem
    - Using the inference to make the classification or regression decision

# Decision Problem - Classification

- Minimizing the mis-classification error
- Decision based on maximum posteriors

$$argmax_j \; p(C_j|\mathbf{x})$$

- Loss matrix
  - Minimizing the expected loss

$$argmax_j \; \sum_k L_{k,j} p(C_k|\mathbf{x})$$

# Approaches for Inference and Decision

I. Finding the joint density from the data.

$$p(C_k|\mathbf{x}) \; \alpha \; p(\mathbf{x}|C_k)p(C_k)$$

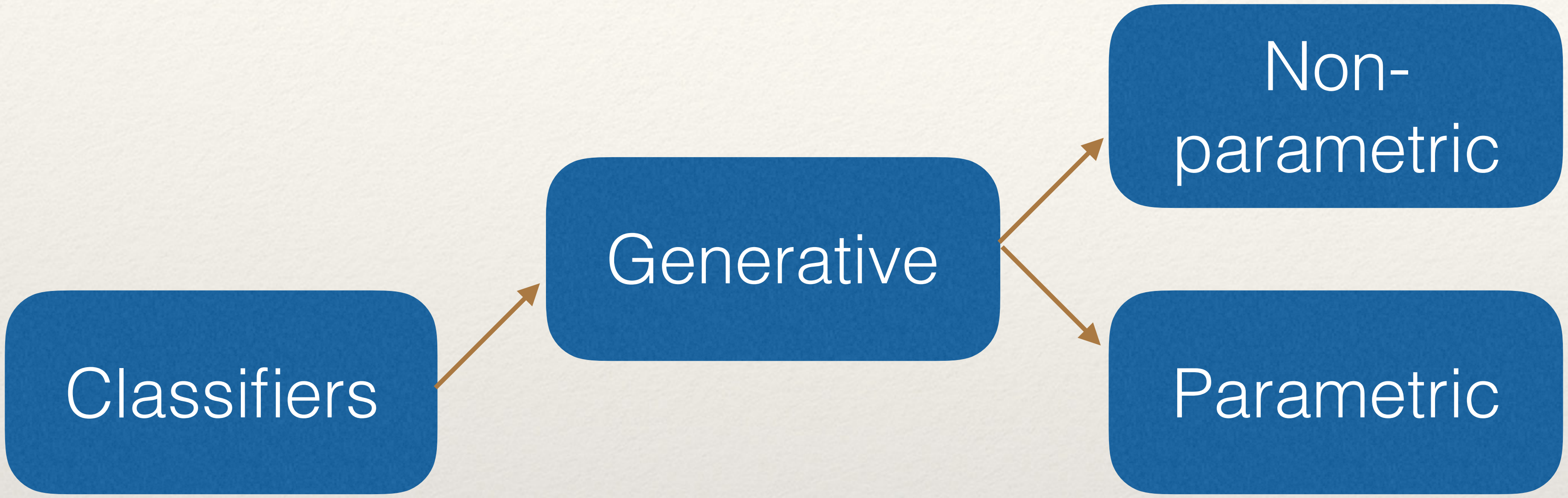
II. Finding the posteriors directly.

III. Using discriminant functions for classification.

# Decision Rule for Regression

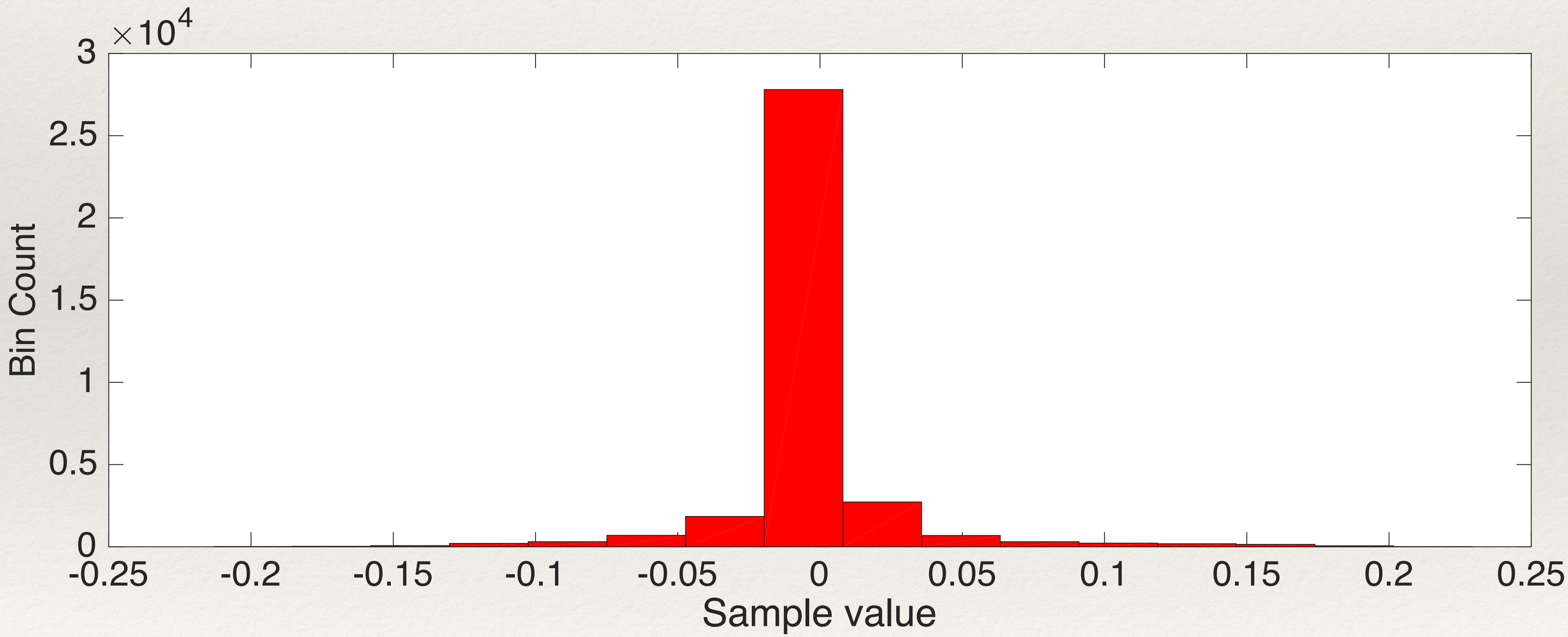
- Minimum mean square error loss
- Solution is conditional expectation.

# Generative Modeling



# Non-parametric Modeling

- Non-parametric models do not specify an apriori set of parameters to model the distribution. Example - Histogram

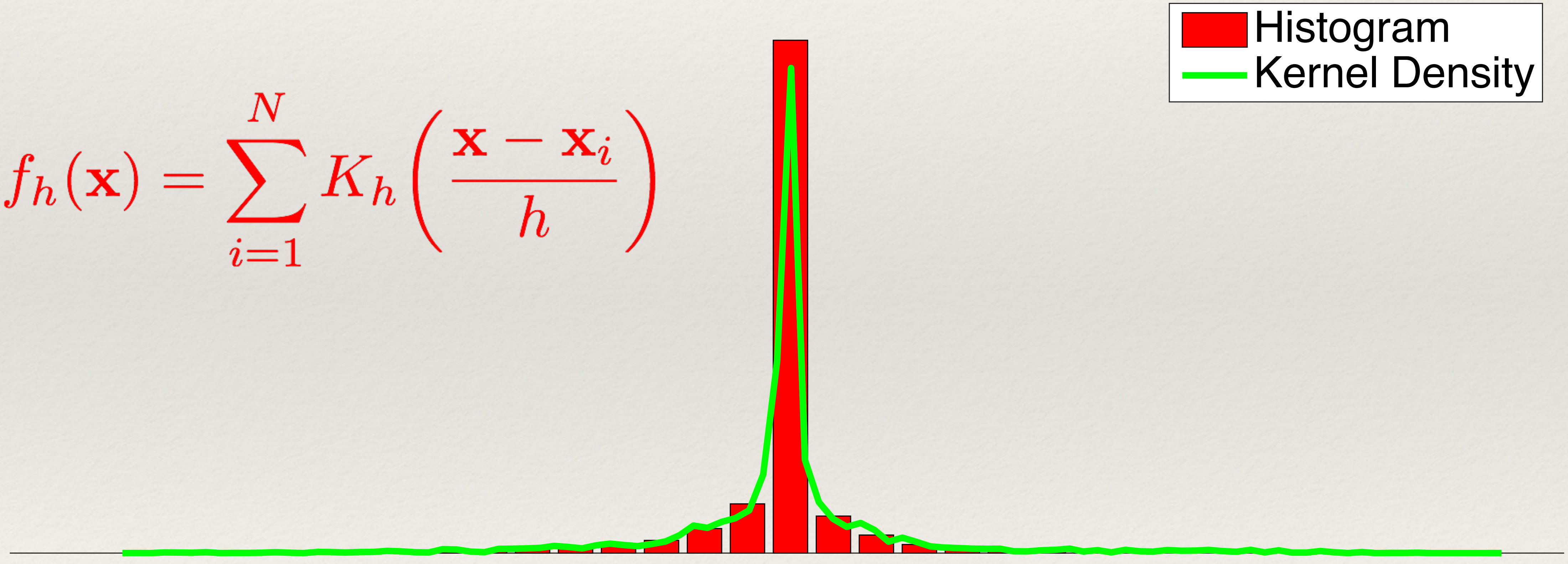



The density is not smooth and has block like shape.

# Non-parametric Modeling

- Non-parametric models do not specify an apriori set of parameters to model the distribution.
    - Example - Kernel Density Estimators

$$f_h(\mathbf{x}) = \sum_{i=1}^{N} K_h\left(\frac{\mathbf{x} - \mathbf{x}_i}{h}\right)$$




Kernel is a smooth function which obeys certain properties

# Non-parametric Modeling

- Non-parametric methods are dependent on number of data points
  - Estimation is difficult for large datasets.
- Likelihood computation and model comparisons are hard.
- Limited use in classifiers

# Parametric Models

- Collection of probability distributions which are described by a finite dimensional parameter set

$$\boldsymbol{\theta} = (\theta_1, \theta_2, ...\theta_K) \qquad P = \{P_{\boldsymbol{\theta}}\}$$

- Examples -
  - Poisson Distribution

$$p_\lambda(j) = \frac{\lambda^j}{j!} e^{-\lambda}$$

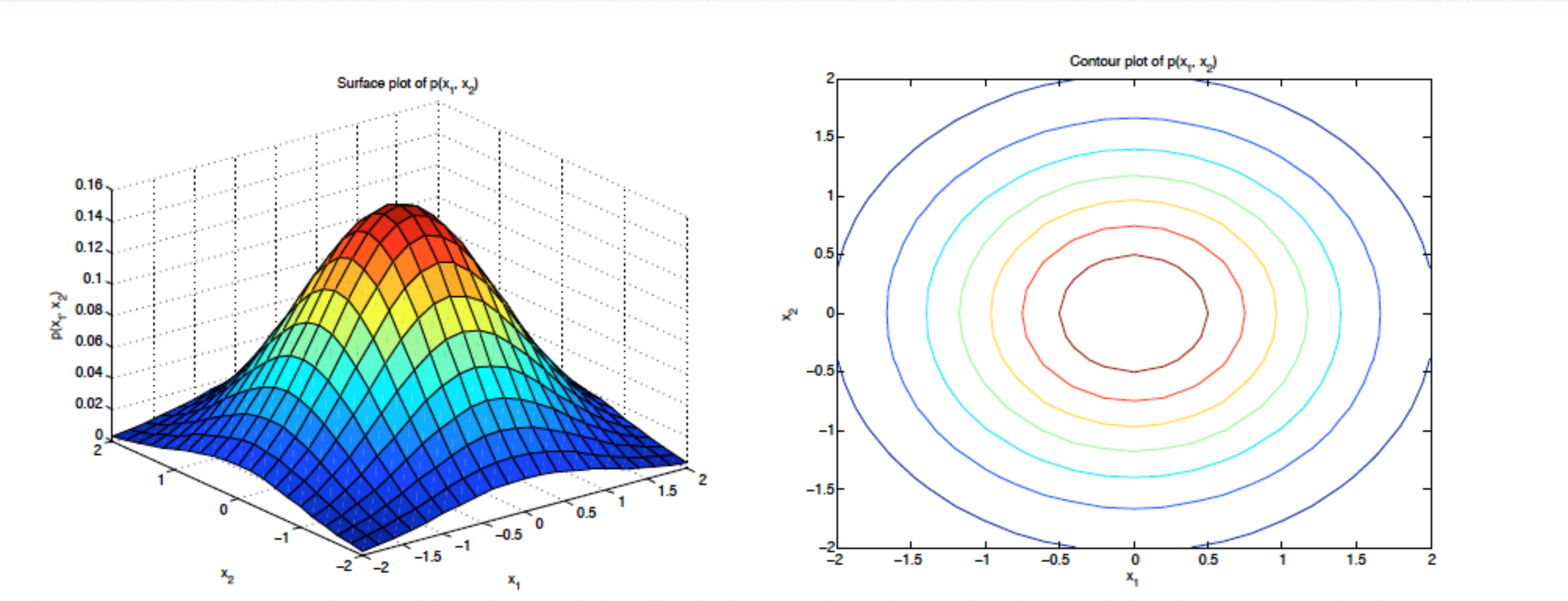
  - Bernoulli Distribution

$$p(\mathbf{x}|\boldsymbol{\mu}) = \prod_{i=1}^{D} \mu_i^{x_i} (1 - \mu_i)^{x_i}$$

  - Gaussian Distribution

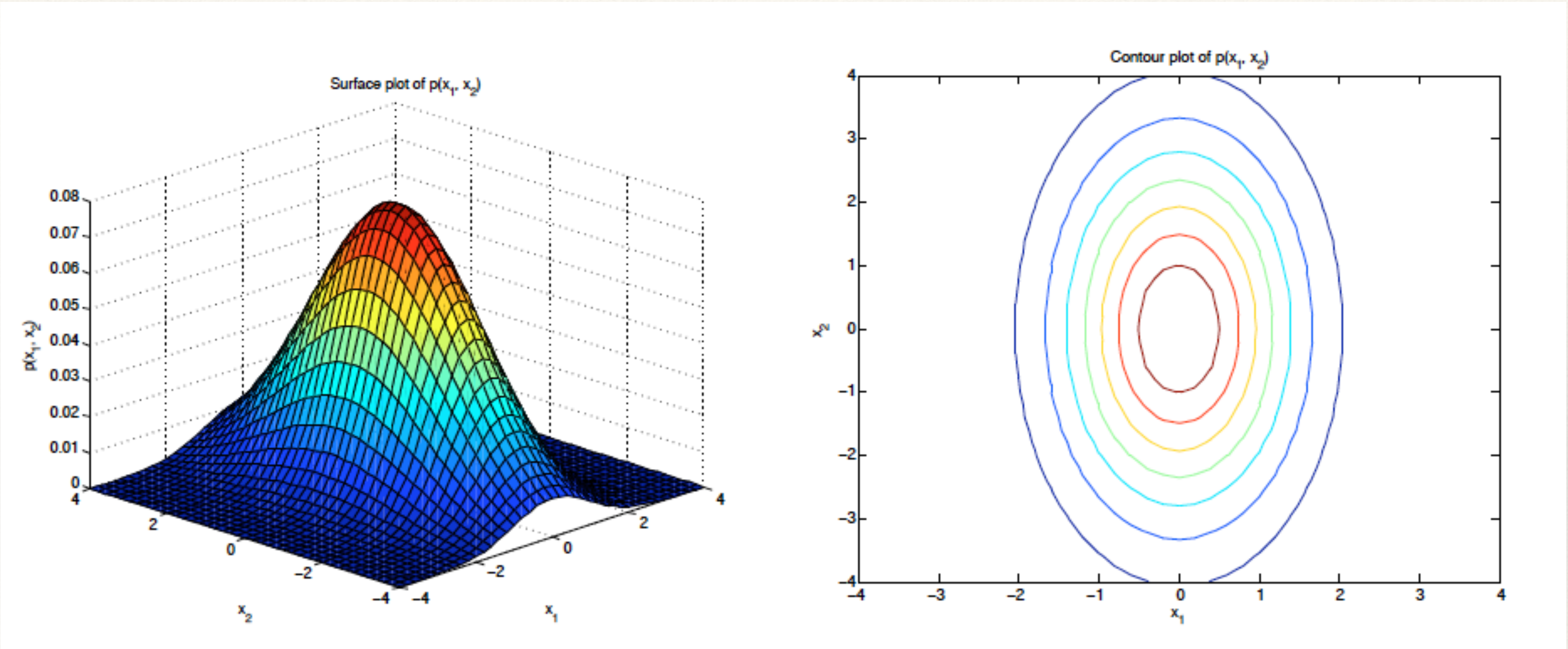
$$p(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^D |\boldsymbol{\Sigma}|}} exp\left\{ -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^* \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\}$$

# Gaussian Distribution



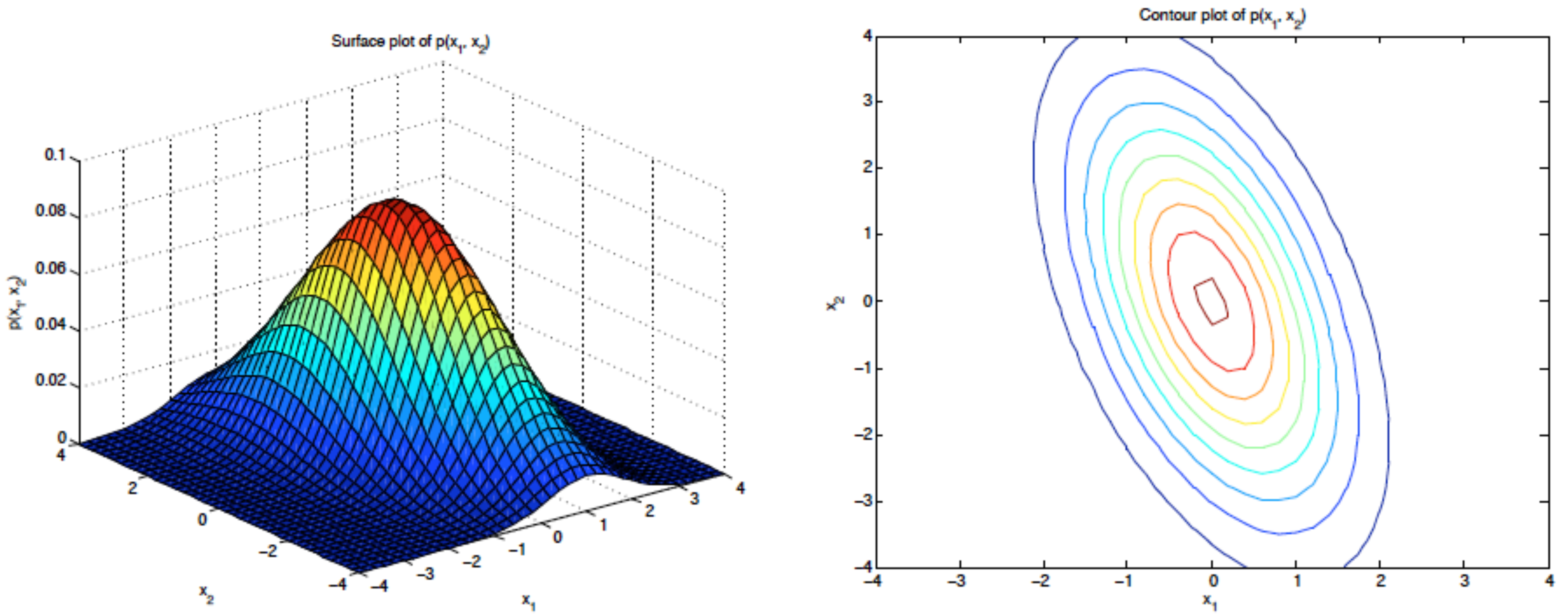

Points of equal probability lie on on contour
Diagonal Gaussian with Identical Variance

# Gaussian Distribution



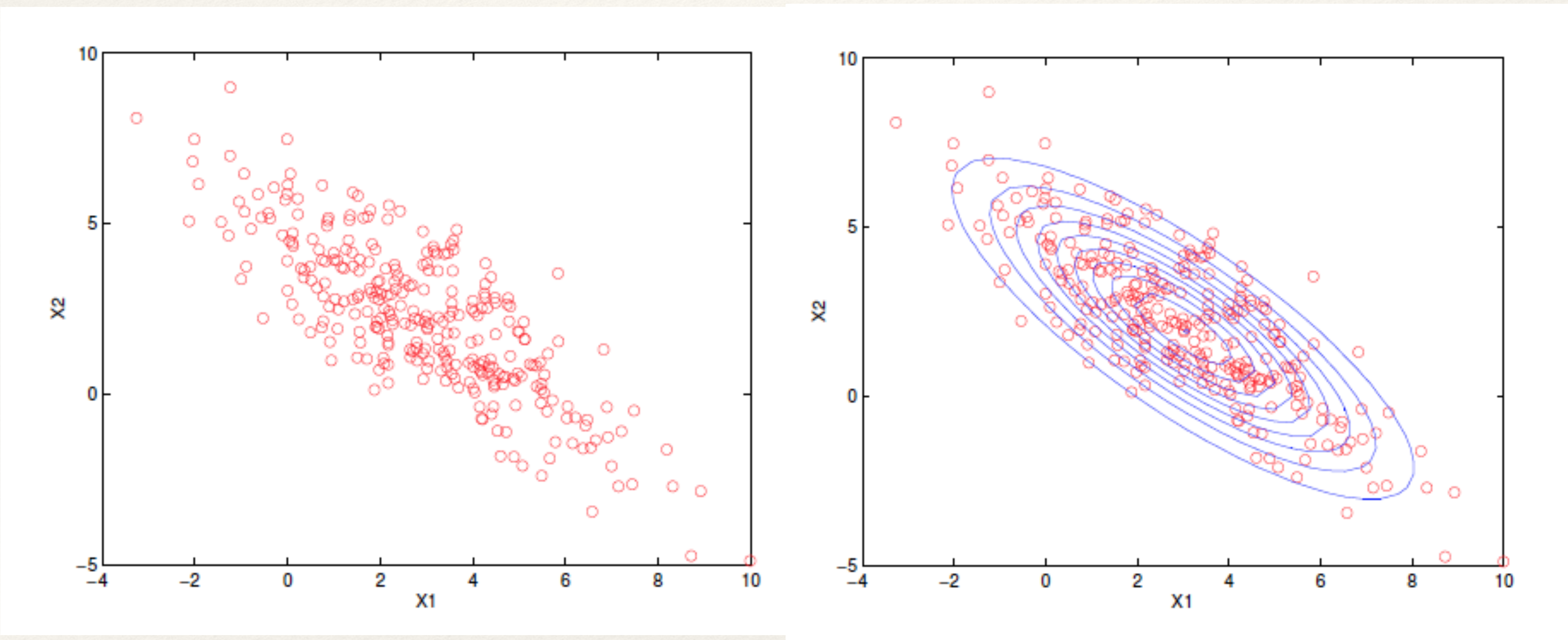

Diagonal Gaussian with different variance

# Gaussian Distribution




Full covariance Gaussian distribution

# Gaussian Distribution



# Finding the parameters of the Model

- The Gaussian model has the following parameters

$$\boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- Total number of parameters to be learned for D dimensional data is $D^2 + D$
- Given N data points $\{\mathbf{x}_i\}_{i=1}^{N}$ how do we estimate the parameters of model.
    - Several criteria can be used
    - The most popular method is the maximum likelihood estimation (MLE).

# MLE

Define the likelihood function as $L(\boldsymbol{\theta}) = \prod_{i=1}^{N} p(\mathbf{x}_i|\boldsymbol{\theta})$

The maximum likelihood estimator (MLE) is

$$\boldsymbol{\theta}^* = arg \max_{\boldsymbol{\theta}} L(\boldsymbol{\theta})$$

The MLE satisfies nice properties like

- Consistency (covergence to true value)
- Efficiency (has the least Mean squared error).

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# MLE

For the Gaussian distribution

$$p(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^D |\boldsymbol{\Sigma}|}} exp\left\{ -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^* \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) \right\}$$

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{N} p(\mathbf{x}_i|\boldsymbol{\theta})$$

$$\log L(\boldsymbol{\theta}) = -\frac{ND}{2} - \frac{N}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}\sum_{i=1}^{N}\left( (\mathbf{x}_i - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) \right)$$

To estimate the parameters $\frac{\partial \log L}{\partial \boldsymbol{\mu}} = 0$

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