MACHINE LEARNING FOR SIGNAL PROCESSING

12-2-2025

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STORY SO FAR

EM algorithm

Generative Modeling

Gaussian Modeling

Gaussian Mixture Modeling

Classification Problem

Decision Theory

Function Modeling

Linear Models for Regression and Classification

Kernel Machines & Max-margin classifiers

Gradient Descent Algorithm

Discriminative Modeling

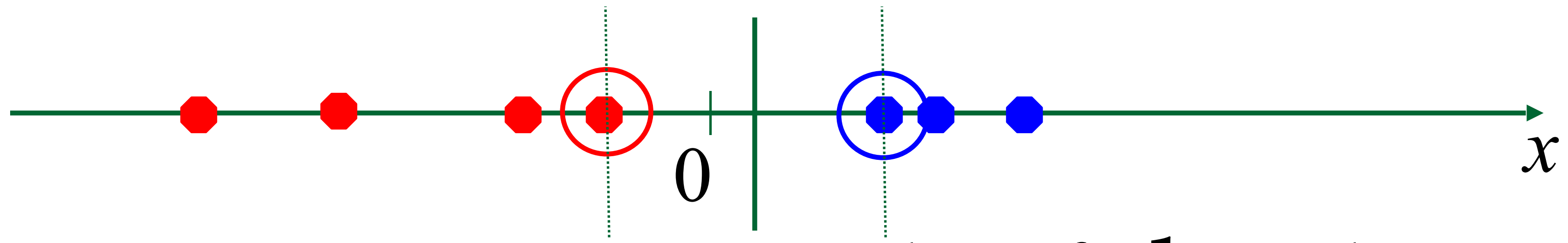
Data Representations PCA, LDA

LINEAR REGRESSION REVISITED

- Primal and dual forms
- Solution in dual space
- Kernels

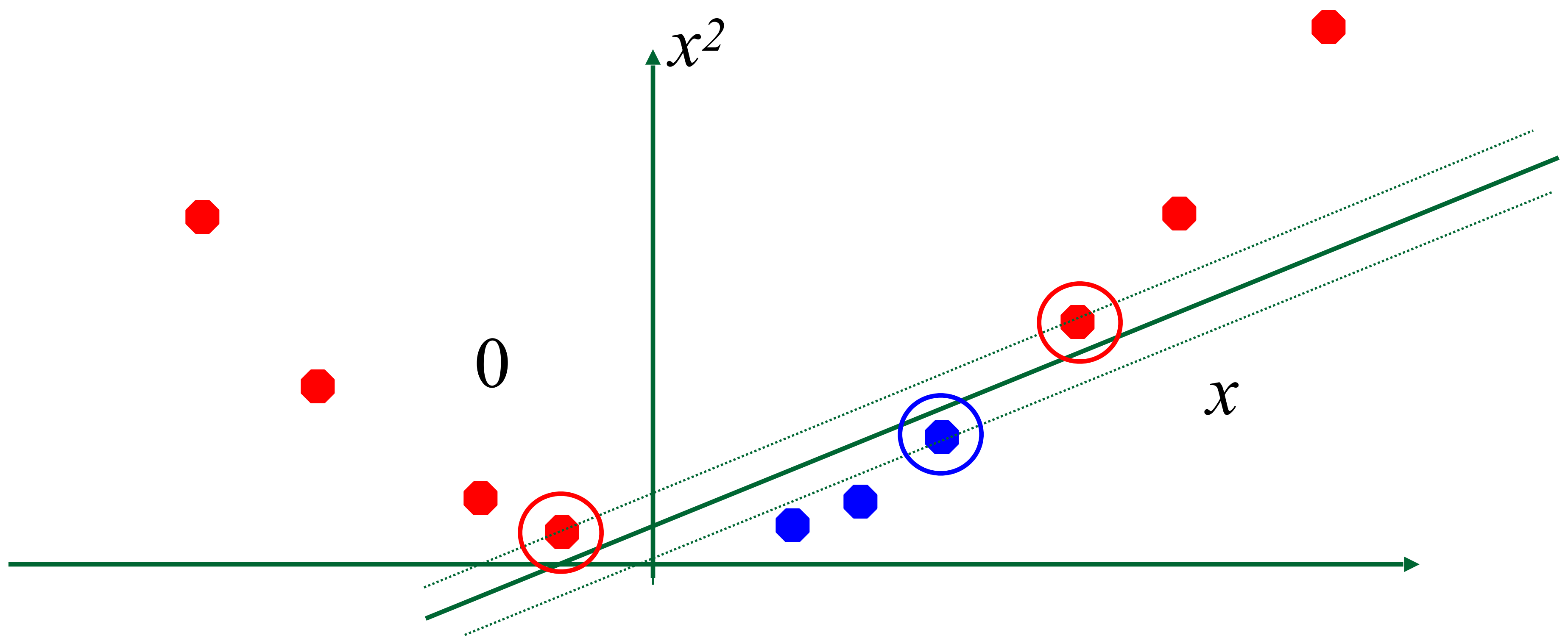
Kernel Machines

- Datasets that are linearly separable with some noise work out great:



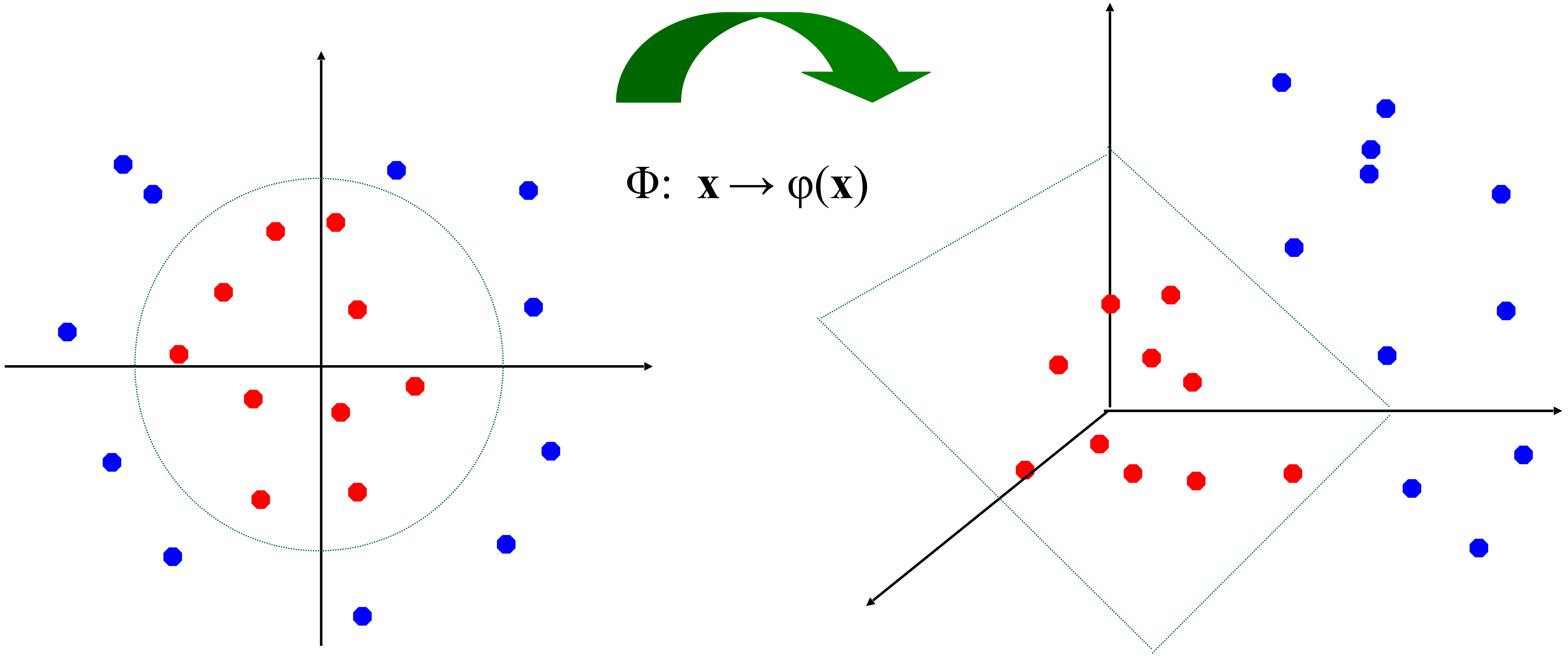
- But what are we going to do if the dataset is just too hard?

- How about… mapping data to a higher-dimensional space:



Kernel Trick

- General idea: the original input space can always be mapped to some higher-dimensional feature space where the training set is separable:



The "Kernel Trick"

- The linear classifier relies on dot product between vectors $k(x_i,x_j)=\mathbf{x}_i^T\mathbf{x}_j$
- If every data point is mapped into high-dimensional space via some transformation Φ: $\mathbf{x} \rightarrow \phi(\mathbf{x})$, the dot product becomes:

$$k(\mathbf{x}_i,\mathbf{x}_j)= \phi(\mathbf{x}_i)^T\phi(\mathbf{x}_j)$$

- A *kernel function* is some function that corresponds to an inner product in some expanded feature space.
- Example:

2-dimensional vectors x=$[x_1 \;\; x_2]$; let $k(\mathbf{x}_i,\mathbf{x}_j)=(1 + \mathbf{x}_i^T\mathbf{x}_j)^2$,

Need to show that $K(\mathbf{x}_i,\mathbf{x}_j)= \boldsymbol{\phi}(\mathbf{x}_i)^T\boldsymbol{\phi}(\mathbf{x}_j)$:

$$k(\mathbf{x}_i,\mathbf{x}_j)=(1 + \mathbf{x}_i^T\mathbf{x}_j)^2,$$

$$= 1+ x_{i1}^2x_{j1}^2 + 2\, x_{i1}x_{j1}\, x_{i2}x_{j2}+ x_{i2}^2x_{j2}^2 + 2x_{i1}x_{j1} + 2x_{i2}x_{j2}$$

$$= [1 \;\; x_{i1}^2 \;\; \sqrt{2}\, x_{i1}x_{i2} \;\; x_{i2}^2 \;\; \sqrt{2}x_{i1} \;\; \sqrt{2}x_{i2}]^T [1 \;\; x_{j1}^2 \;\; \sqrt{2}\, x_{j1}x_{j2} \;\; x_{j2}^2 \;\; \sqrt{2}x_{j1} \;\; \sqrt{2}x_{j2}]$$

$$= \boldsymbol{\phi}(\mathbf{x}_i)^T\boldsymbol{\phi}(\mathbf{x}_j), \quad \text{where } \boldsymbol{\phi}(\mathbf{x}) = [1 \;\; x_1^2 \;\; \sqrt{2}\, x_1x_2 \;\; x_2^2 \;\; \sqrt{2}x_1 \;\; \sqrt{2}x_2]$$

KERNELS

- For many functions $k(\mathbf{x_i},\mathbf{x_j})$ checking that

  $k(\mathbf{x_i},\mathbf{x_j}) = \phi(\mathbf{x_i})^T\phi(\mathbf{x_j})$ can be cumbersome.
- Mercer's theorem: Every semi-positive definite symmetric function is a kernel
  - Semi-positive definite symmetric functions correspond to a semi-positive definite symmetric Gram matrix:

K =

$k(\mathbf{x}_1,\mathbf{x}_1)$	$k(\mathbf{x}_1,\mathbf{x}_2)$	$k(\mathbf{x}_1,\mathbf{x}_3)$	…	$k(\mathbf{x}_1,\mathbf{x}_N)$
$k(\mathbf{x}_2,\mathbf{x}_1)$	$k(\mathbf{x}_2,\mathbf{x}_2)$	$k(\mathbf{x}_2,\mathbf{x}_3)$		$k(\mathbf{x}_2,\mathbf{x}_N)$
…	…	…	…	…
$k(\mathbf{x}_N,\mathbf{x}_1)$	$k(\mathbf{x}_N,\mathbf{x}_2)$	$k(\mathbf{x}_N,\mathbf{x}_3)$	…	$k(\mathbf{x}_N,\mathbf{x}_N)$

EXAMPLES OF KERNEL FUNCTIONS

- Linear: $k(\mathbf{x}_i,\mathbf{x}_j) = \mathbf{x}_i{}^T\mathbf{x}_j$
- Polynomial of power p: $k(\mathbf{x}_i,\mathbf{x}_j) = (1 + \mathbf{x}_i{}^T\mathbf{x}_j)^p$
- Gaussian (radial-basis function network):

$$k(\mathbf{x}_i, \mathbf{x}_j) = \exp \frac{-||\mathbf{x}_i - \mathbf{x}_j||^2}{\sigma^2}$$

- Sigmoid: $k(\mathbf{x}_i,\mathbf{x}_j) = \tanh(\beta_0\mathbf{x}_i{}^T\mathbf{x}_j + \beta_1)$

Properties of Kernel Functions

$$\begin{aligned}
k(\mathbf{x},\mathbf{x}') &= ck_1(\mathbf{x},\mathbf{x}') \\
k(\mathbf{x},\mathbf{x}') &= f(\mathbf{x})k_1(\mathbf{x},\mathbf{x}')f(\mathbf{x}') \\
k(\mathbf{x},\mathbf{x}') &= q\left(k_1(\mathbf{x},\mathbf{x}')\right) \\
k(\mathbf{x},\mathbf{x}') &= \exp\left(k_1(\mathbf{x},\mathbf{x}')\right) \\
k(\mathbf{x},\mathbf{x}') &= k_1(\mathbf{x},\mathbf{x}') + k_2(\mathbf{x},\mathbf{x}') \\
k(\mathbf{x},\mathbf{x}') &= k_1(\mathbf{x},\mathbf{x}')k_2(\mathbf{x},\mathbf{x}') \\
k(\mathbf{x},\mathbf{x}') &= k_3\left(\boldsymbol{\phi}(\mathbf{x}),\boldsymbol{\phi}(\mathbf{x}')\right) \\
k(\mathbf{x},\mathbf{x}') &= \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}' \\
k(\mathbf{x},\mathbf{x}') &= k_a(\mathbf{x}_a,\mathbf{x}'_a) + k_b(\mathbf{x}_b,\mathbf{x}'_b) \\
k(\mathbf{x},\mathbf{x}') &= k_a(\mathbf{x}_a,\mathbf{x}'_a)k_b(\mathbf{x}_b,\mathbf{x}'_b)
\end{aligned}$$

Non-linear Kernel Function

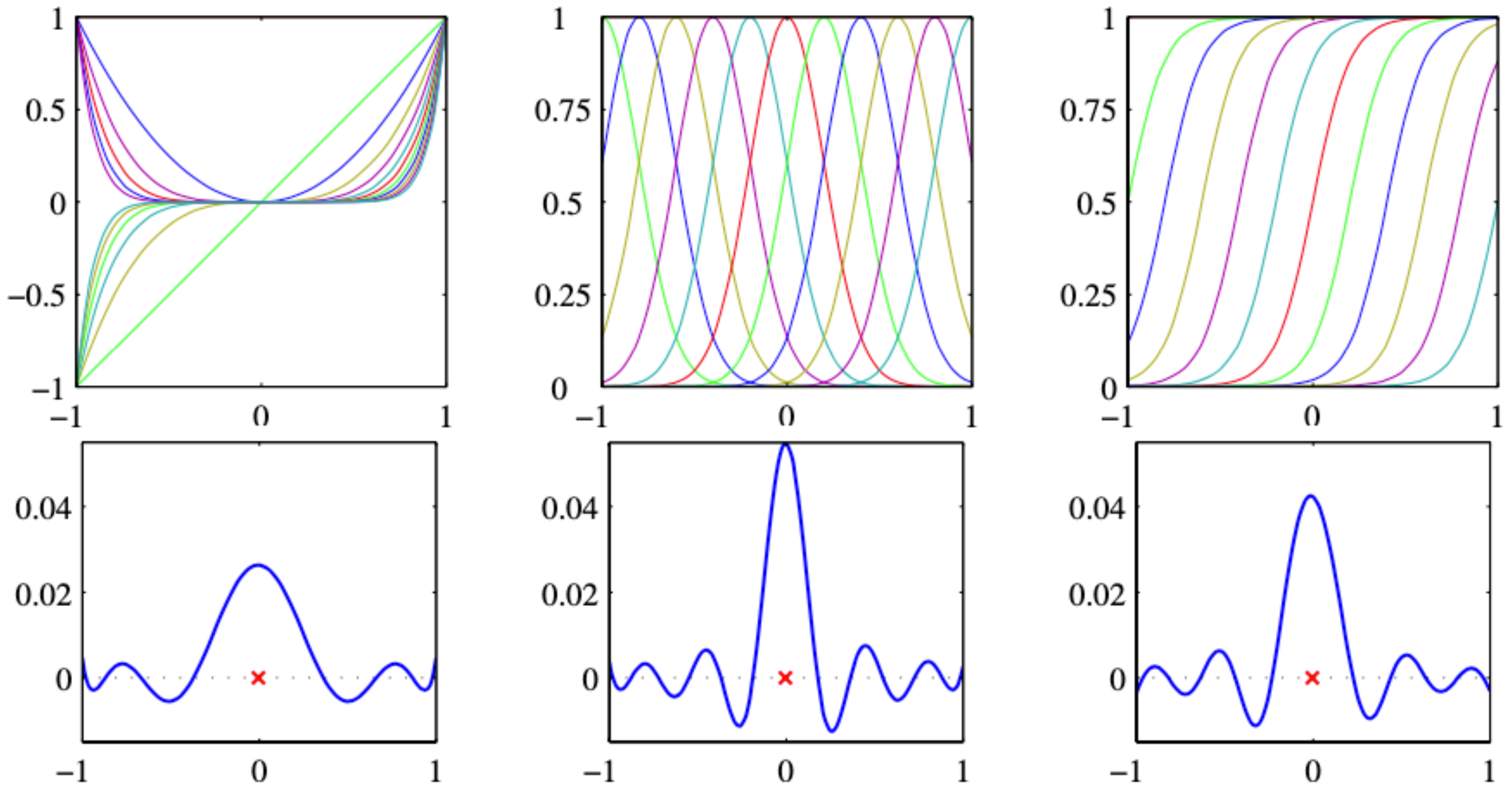
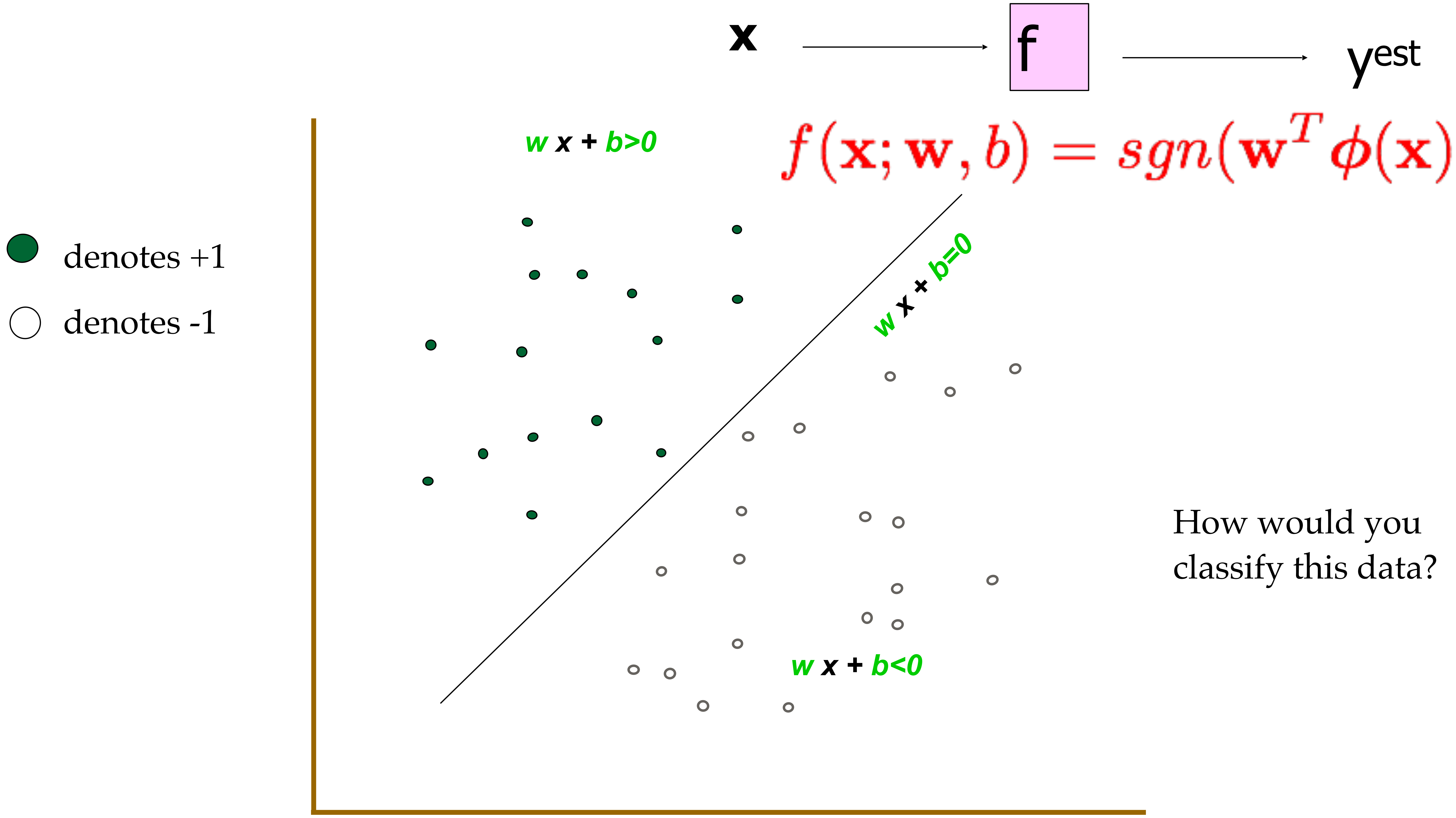


Figure 6.1 Illustration of the construction of kernel functions starting from a corresponding set of basis functions. In each column the lower plot shows the kernel function $k(x, x')$ defined by (6.10) plotted as a function of x for $x' = 0$, while the upper plot shows the corresponding basis functions given by polynomials (left column), 'Gaussians' (centre column), and logistic sigmoids (right column).

Linear Classifiers



$$f(\mathbf{x}; \mathbf{w}, b) = sgn(\mathbf{w}^T \phi(\mathbf{x}) + b)$$

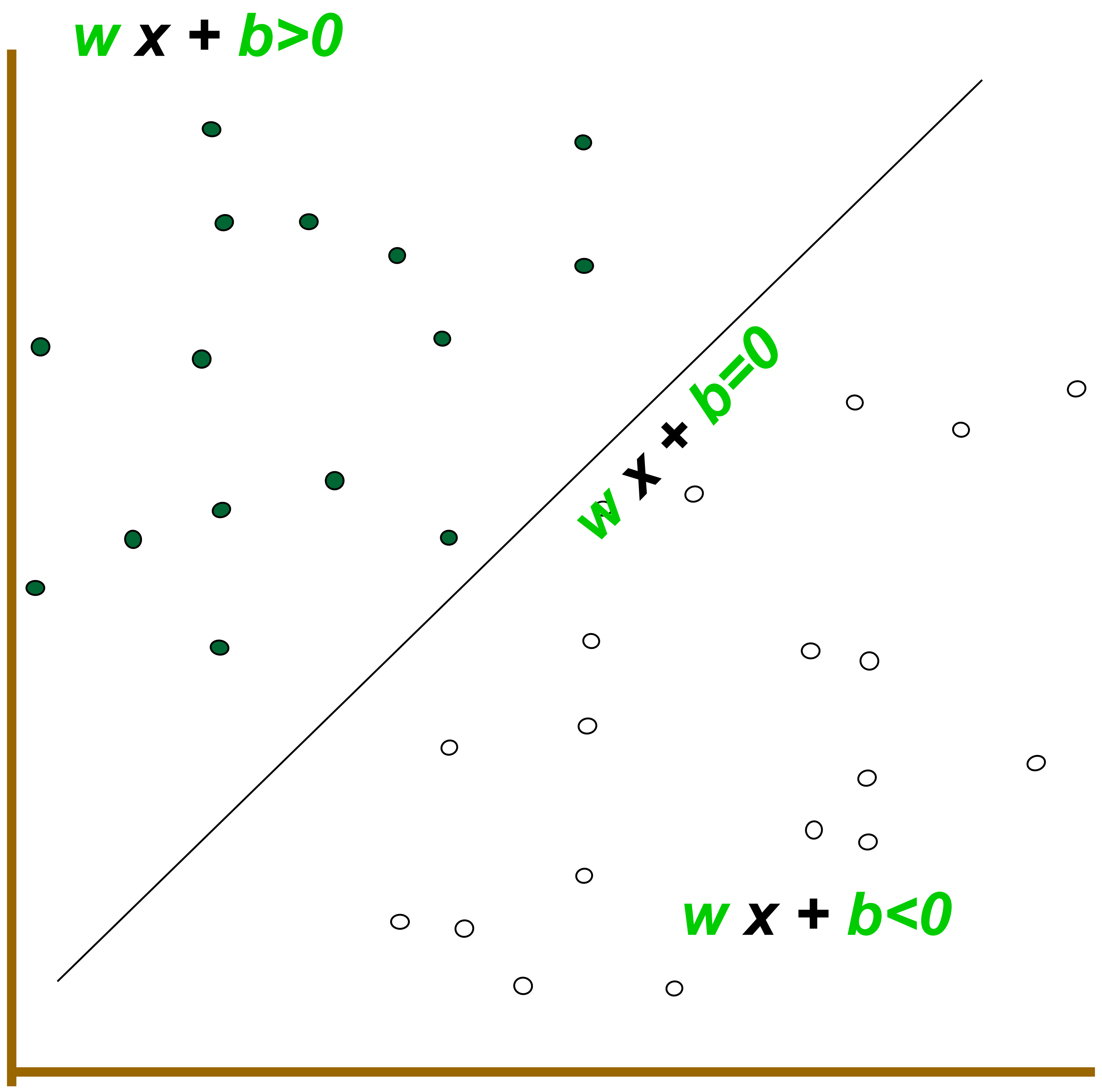
How would you classify this data?

Linear Classifiers

$$\mathbf{x} \longrightarrow \boxed{f} \longrightarrow y^{est}$$

$$f(\mathbf{x}; \mathbf{w}, b) = sgn(\mathbf{w}^T \phi(\mathbf{x}) + b)$$

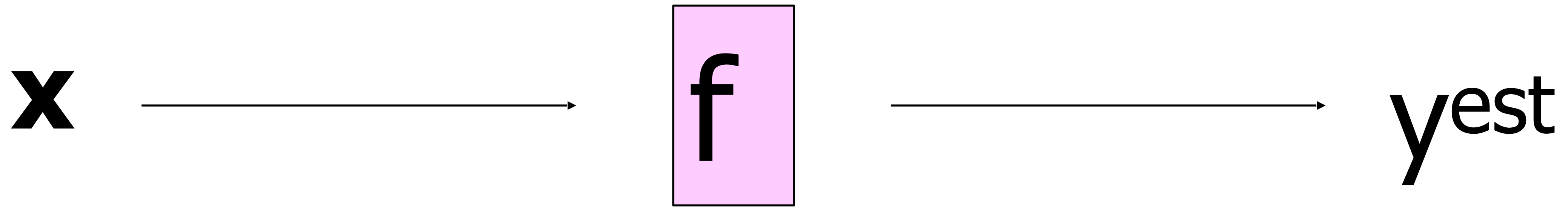
- denotes +1
- denotes -1



How would you classify this data?

"SVM and applications", Mingyue Tan. Univ of British Columbia

LINEAR CLASSIFIERS



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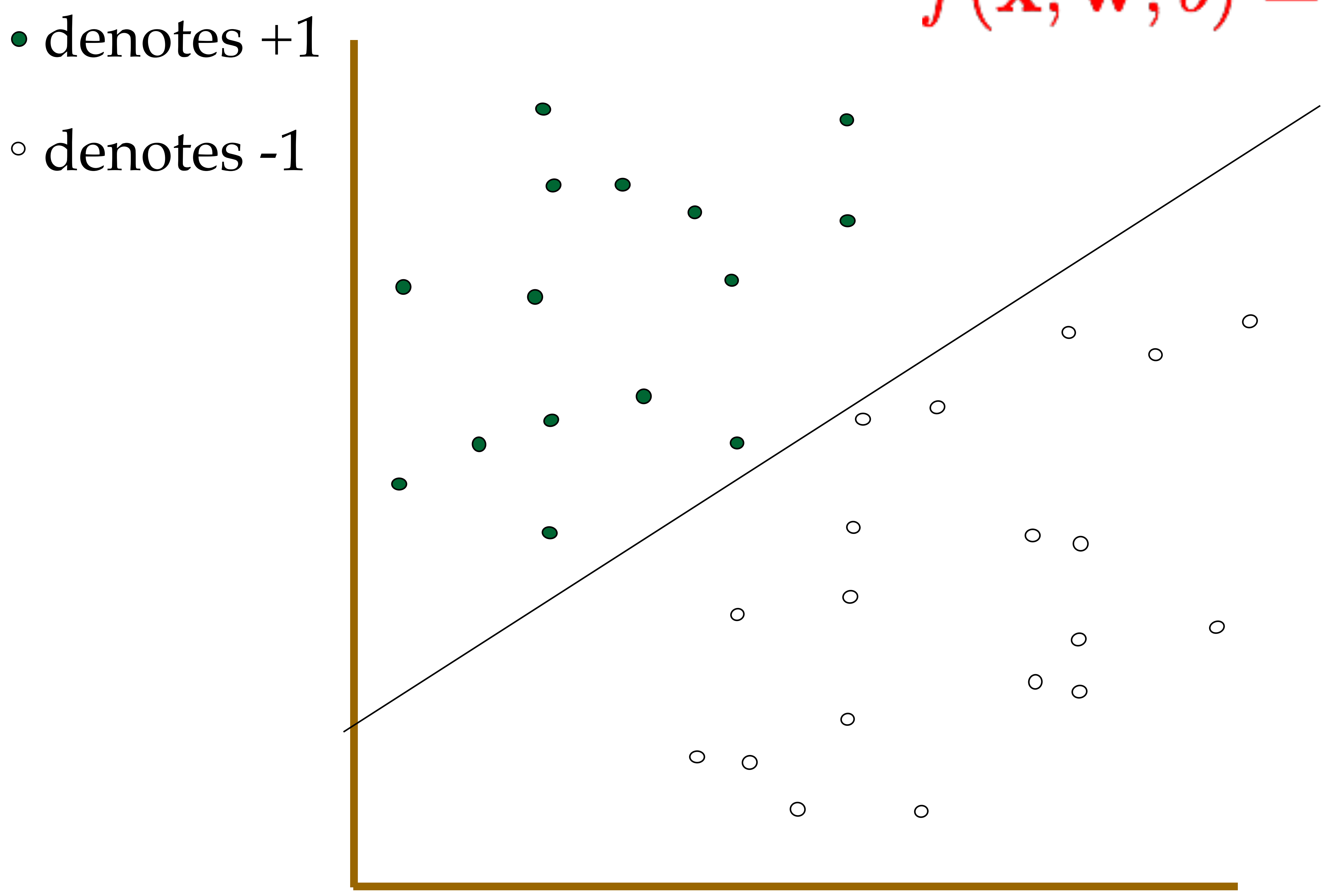

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Linear Classifiers

x ⟶ f ⟶ yest

$$f(\mathbf{x}; \mathbf{w}, b) = sgn(\mathbf{w}^T \phi(\mathbf{x}) + b)$$

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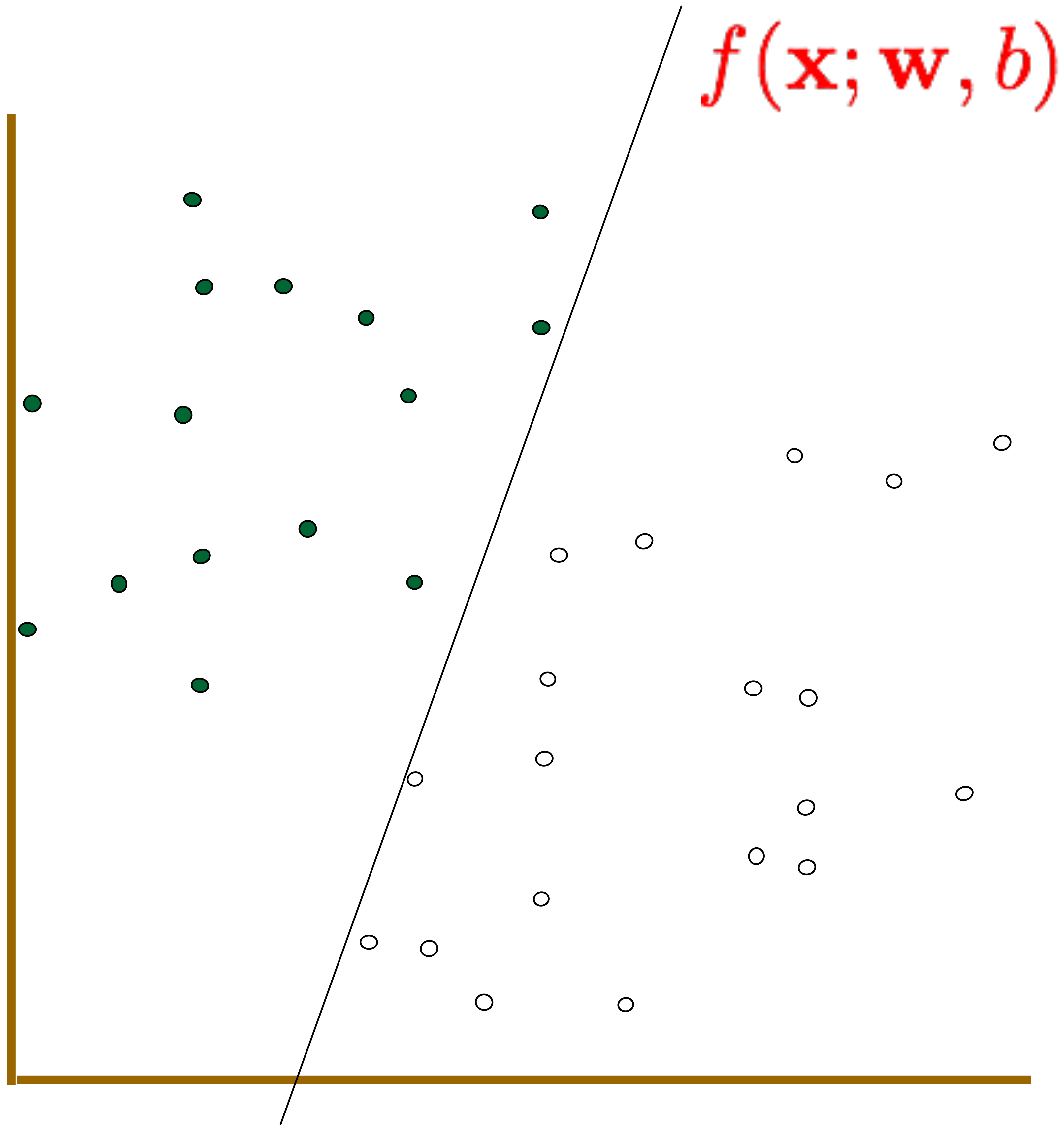

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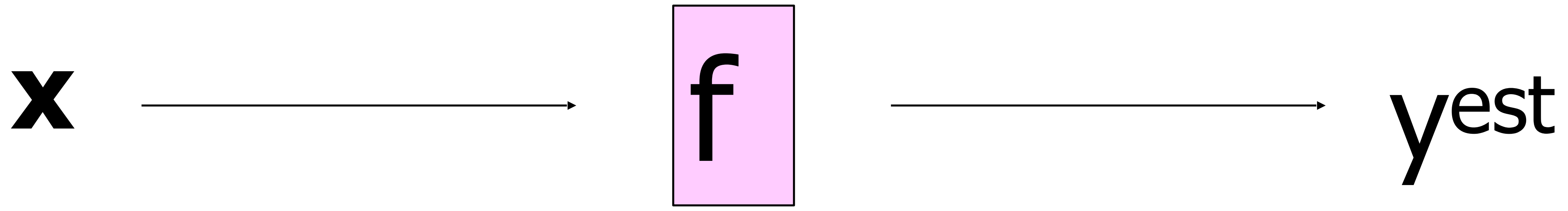
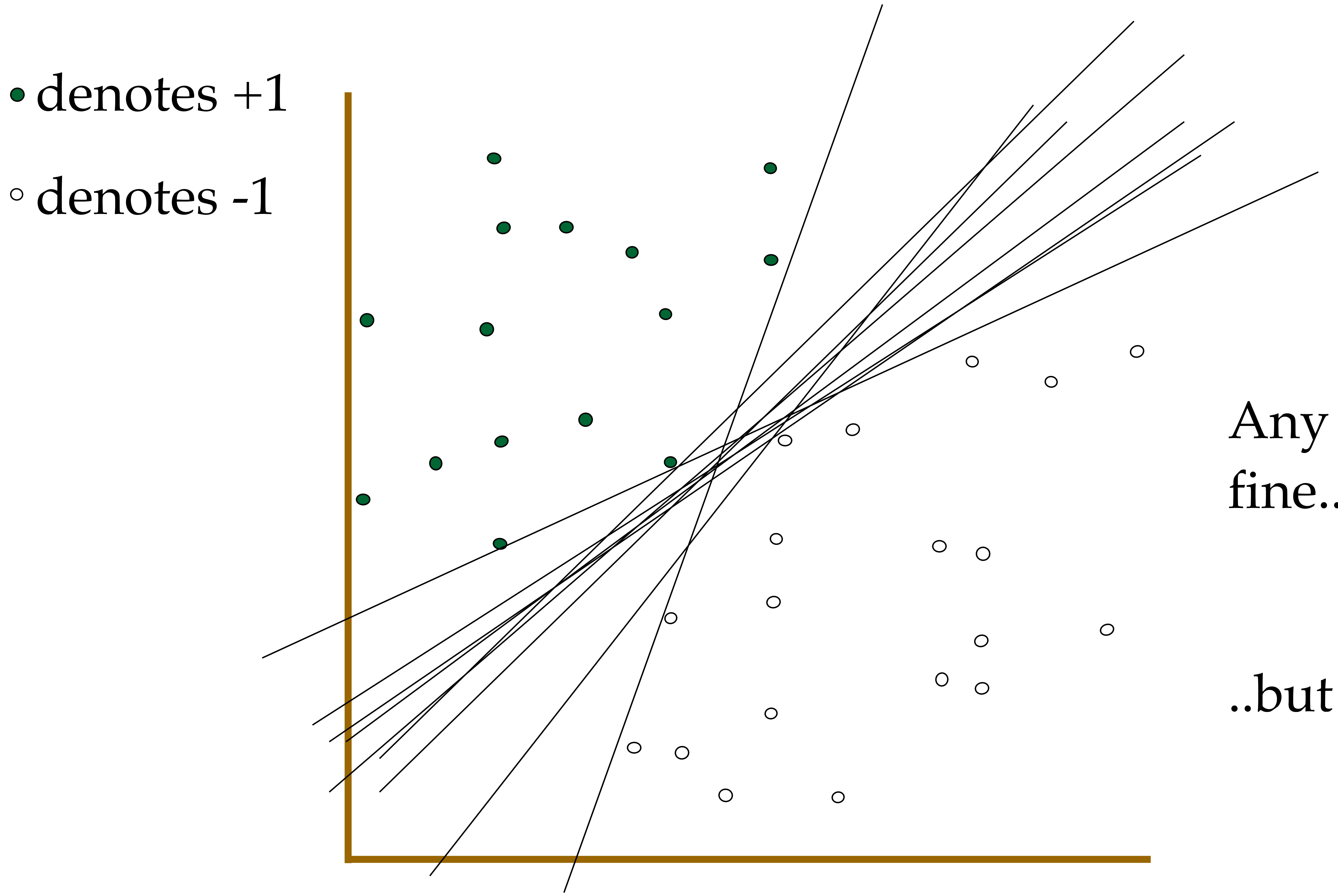
Linear Classifiers





Any of these would be fine..

..but which is best?

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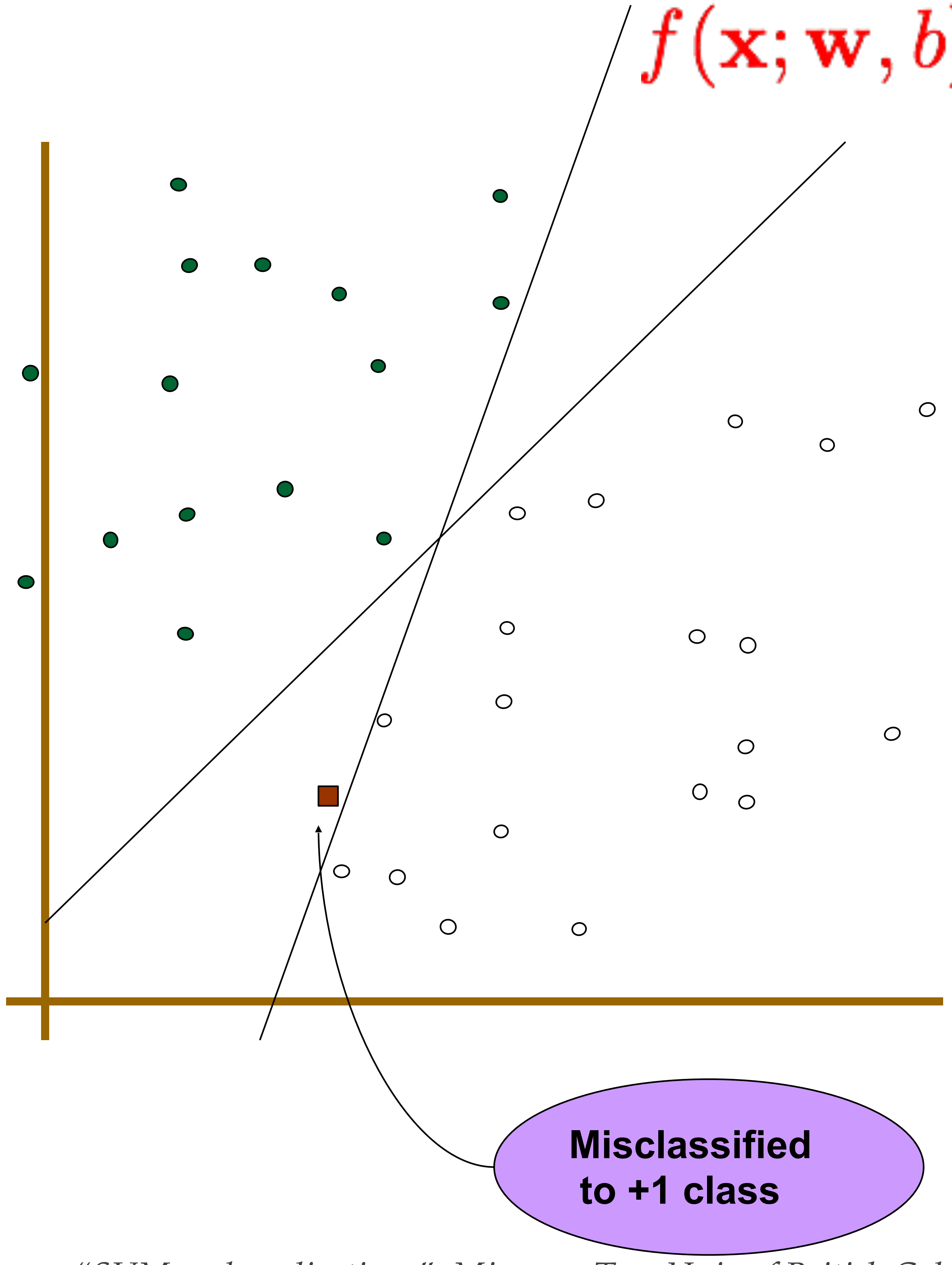
LINEAR CLASSIFIERS

$$\mathbf{x} \longrightarrow \boxed{f} \longrightarrow y^{est}$$

$$f(\mathbf{x}; \mathbf{w}, b) = sgn(\mathbf{w}^T \phi(\mathbf{x}) + b)$$

- ● denotes +1
- ○ denotes -1



How would you classify this data?

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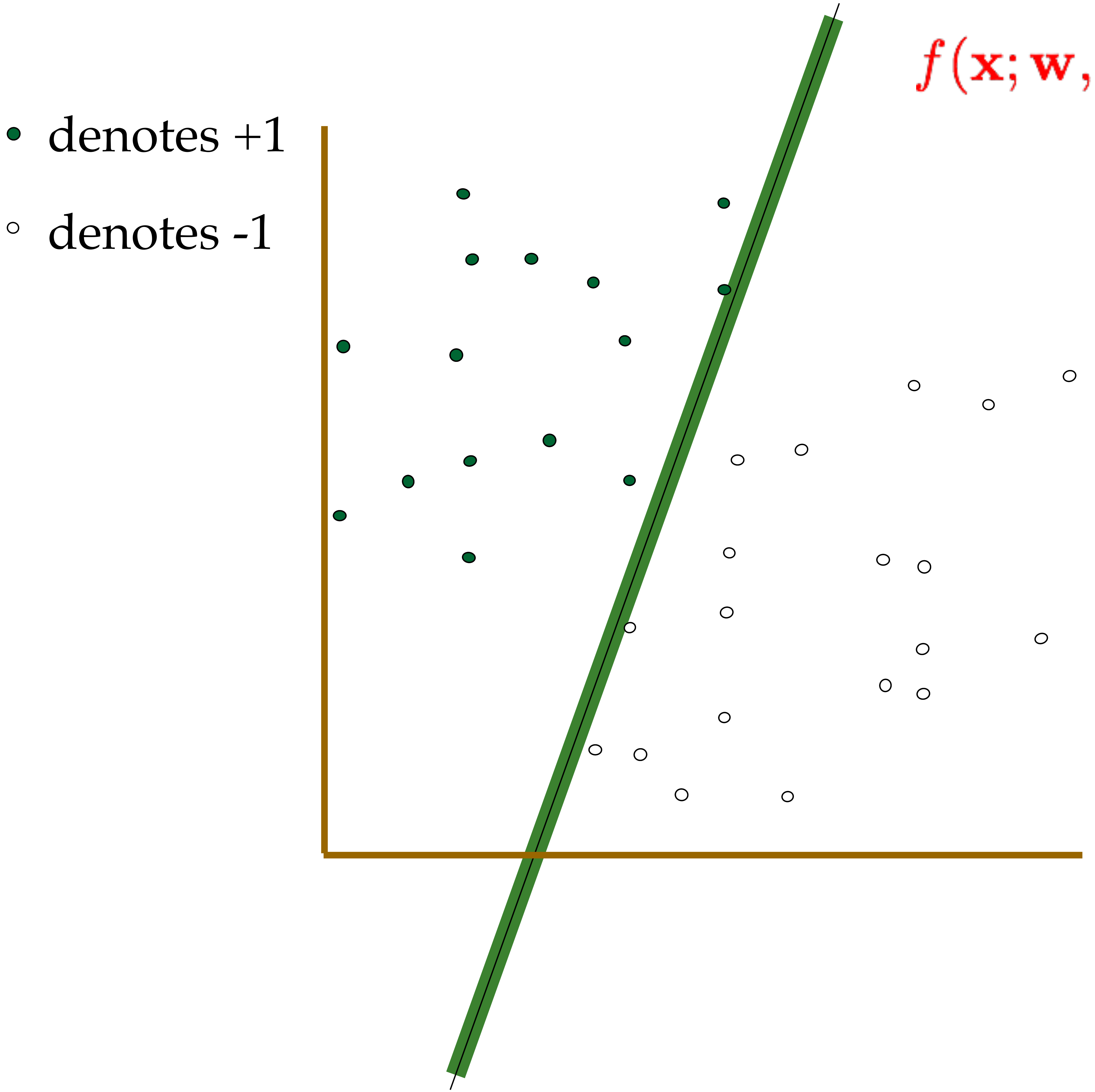
MAX-MARGIN CLASSIFIERS

$\mathbf{x} \longrightarrow \boxed{f} \longrightarrow$ yest

$$f(\mathbf{x}; \mathbf{w}, b) = sgn(\mathbf{w}^T \phi(\mathbf{x}) + b)$$



Define the **margin** of a linear classifier as the width that the boundary could be increased by before hitting a datapoint.

SVM Formulation

- Goal - **1) Correctly classify all training data**

$$\left.\begin{array}{ll}\mathbf{w}^T\phi(\mathbf{x}_n)+b\geq 1 & if \quad t_n=+1\\ \mathbf{w}^T\phi(\mathbf{x}_n)+b\leq 1 & if \quad t_n=-1\end{array}\right\}$$

$$t_n(\mathbf{w}^T\phi(\mathbf{x}_n)+b)\geq 1$$

2) Define the Margin

$$\frac{1}{||\mathbf{w}||}min_n\ [t_n(\mathbf{w}^T\phi(\mathbf{x}_n)+b)]$$

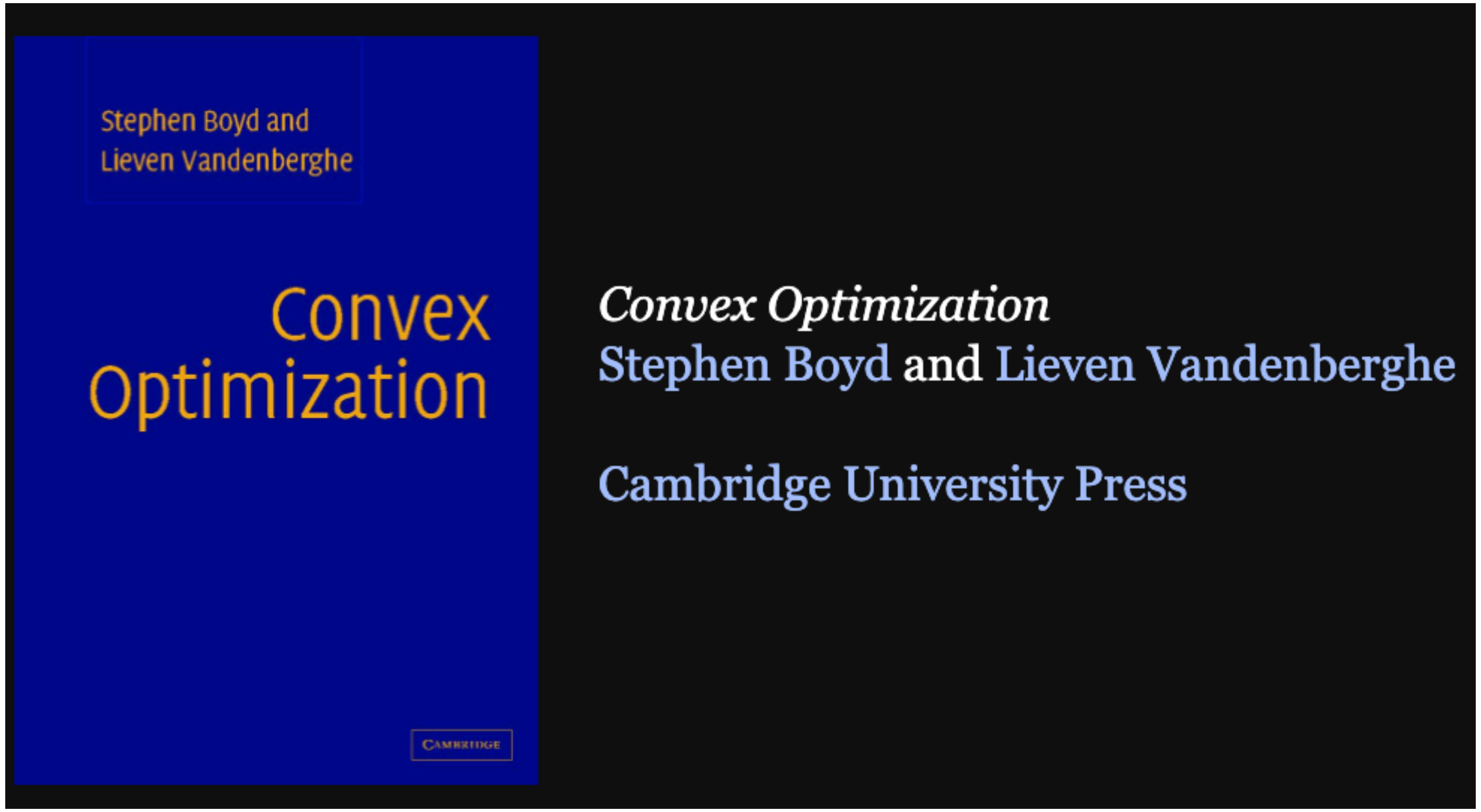
3) Maximize the Margin

$$argmax_{\mathbf{w},b}\left\{\frac{1}{||\mathbf{w}||}min_n\ [t_n(\mathbf{w}^T\phi(\mathbf{x}_n)+b)]\right\}$$

- Equivalently written as

$$argmin_{\mathbf{w},b}\ \frac{1}{2}||\mathbf{w}||^2 \text{such that} \quad t_n(\mathbf{w}^T\phi(\mathbf{x}_n)+b)\geq 1$$

Constrained Optimization Basics



https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf

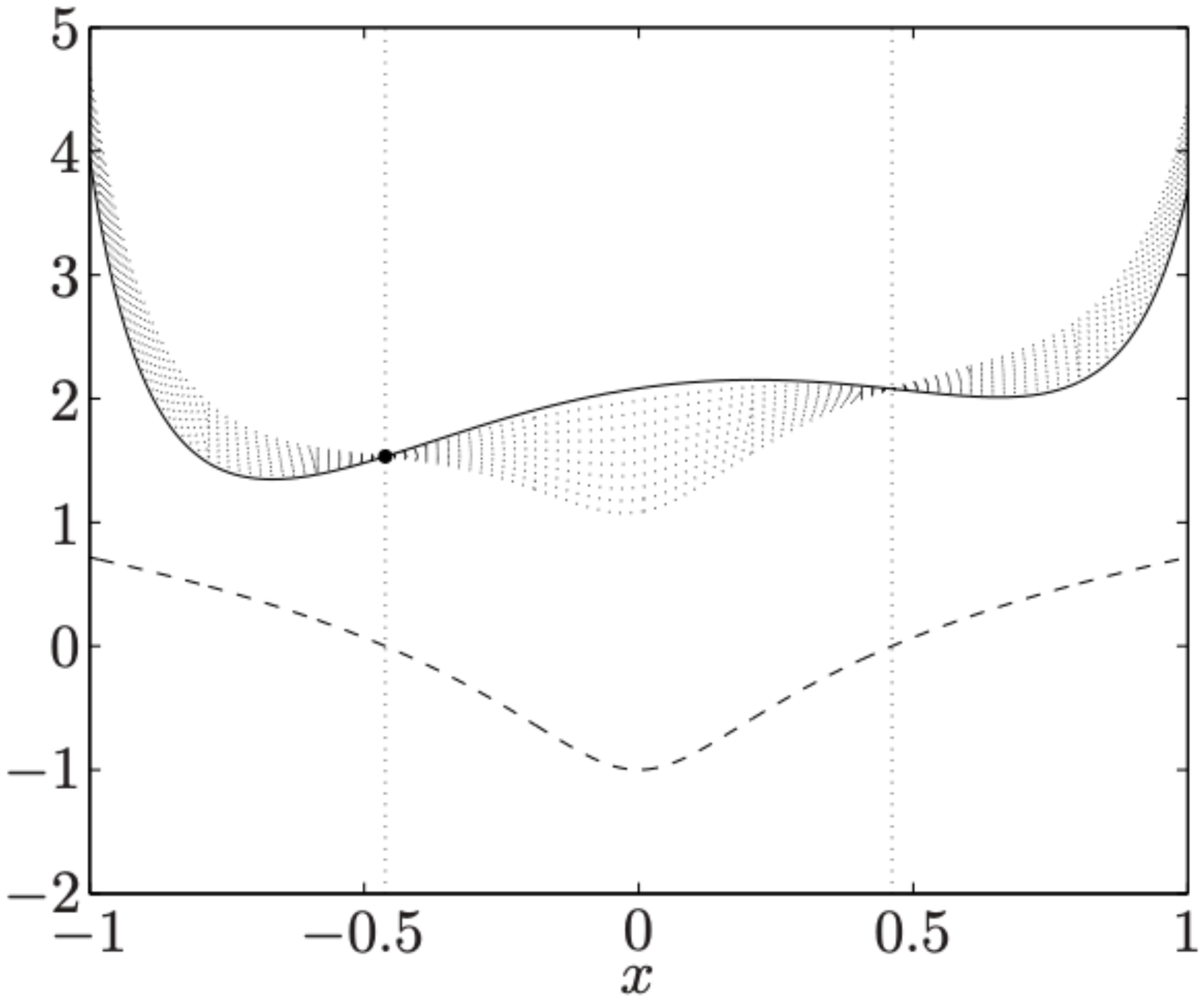


Figure 5.1 *Lower bound from a dual feasible point.* The solid curve shows the objective function f_0, and the dashed curve shows the constraint function f_1. The feasible set is the interval $[-0.46, 0.46]$, which is indicated by the two dotted vertical lines. The optimal point and value are $x^\star = -0.46$, $p^\star = 1.54$ (shown as a circle). The dotted curves show $L(x, \lambda)$ for $\lambda = 0.1,\ 0.2, \ldots, 1.0$. Each of these has a minimum value smaller than $p^\star$, since on the feasible set (and for $\lambda \geq 0$) we have $L(x, \lambda) \leq f_0(x)$.

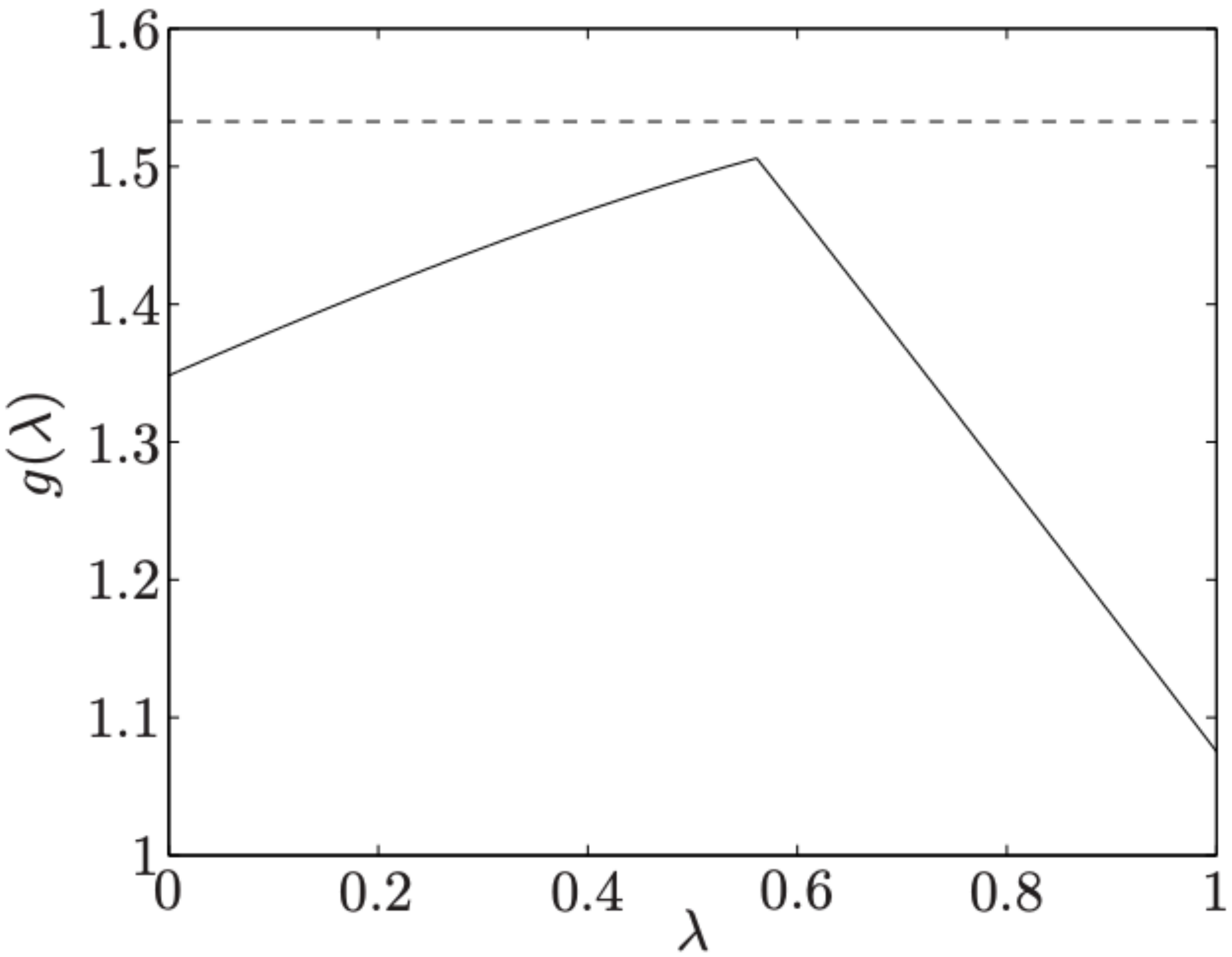


Figure 5.2 The dual function g for the problem in figure 5.1. Neither f_0 nor f_1 is convex, but the dual function is concave. The horizontal dashed line shows $p^\star$, the optimal value of the problem.

Solving the Optimization Problem

- Need to optimize a *quadratic* function subject to *linear* constraints.
- Quadratic optimization problems are a well-known class of mathematical programming problems, and many (rather intricate) algorithms exist for solving them.
- The solution involves constructing a *dual problem* where a *Lagrange multiplier* a_n is associated with every constraint in the primary problem:
- The dual problem in this case is maximized

Find $\{a_1, .., a_N\}$ such that

$$\tilde{L}(\mathbf{a}) = \sum_{n=1}^{N} a_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} t_n t_m a_n a_m k(\mathbf{x}_n, \mathbf{x}_m) \text{ maximized}$$

and $\sum_n a_n t_n = 0, \quad a_n \geq 0$

Solving the Optimization Problem

- The solution has the form:

$$\mathbf{w} = \sum_{n=1}^{N} a_n \phi(\mathbf{x}_n)$$

- Each non-zero a_n indicates that corresponding $\mathbf{x_n}$ is a support vector. Let S denote the set of support vectors.

$$b = y(\mathbf{x}_n) - \sum_{m \in S} a_m k(\mathbf{x}_m, \mathbf{x}_n)$$

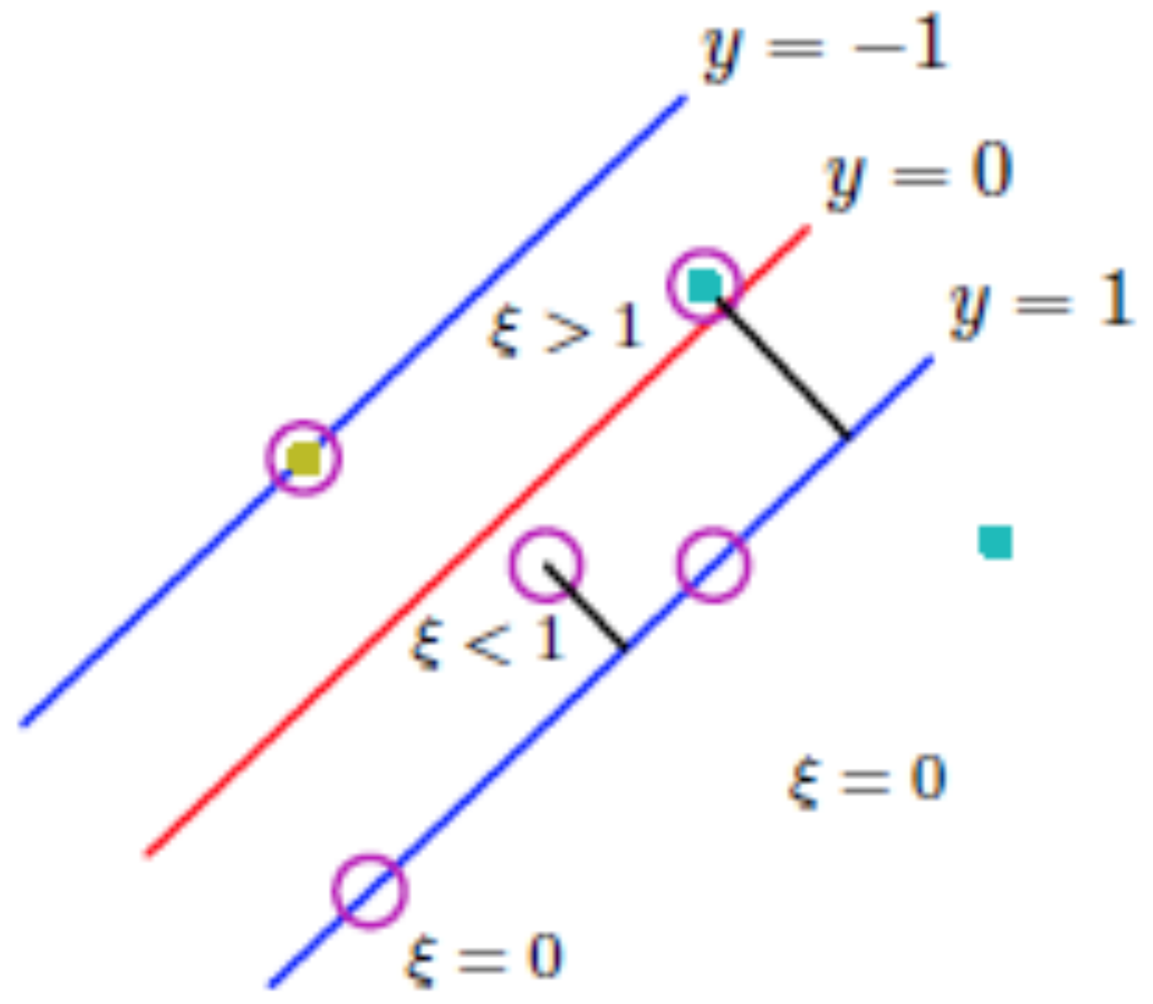
- And the classifying function will have the form:

$$y(\mathbf{x}) = \sum_{n \in S} a_n k(\mathbf{x}_n, \mathbf{x}) + b$$

Overlapping class boundaries

- The classes are not linearly separable - Introducing slack variables ζ_n
- Slack variables are non-negative $\zeta_n \geq 0$
- They are defined using

$$t_n y(\mathbf{x}_n) \geq 1 - \zeta_n$$



- The upper bound on mis-classification

$$\sum_n \zeta_n$$

- The cost function to be optimized in this case

$$C \sum_n \zeta_n + \frac{1}{2} \mathbf{w}^T \mathbf{w}$$

SVM Formulation - overlapping classes

- Formulation very similar to previous case except for additional constraints

$$0 \leq a_n \leq C$$

- Solved using the dual formulation - sequential minimal optimization algorithm
- Final classifier is based on the sign of

$$y(\mathbf{x}) = \sum_{n \in S} a_n k(\mathbf{x}_n, \mathbf{x}) + b$$

THANK YOU

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